ベルヌーイ ディストリビューションについて

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On Bernoulli distributions

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The Bernoulli distribution is a typical example of p-adic distributions, which is constructed from the Bernoulli polynomial. In this note, a characterization of the Bernoulli distribution is given.

Key words: P-adic distributions, P-adic measures

Let X be a topological space. A *p-adic distribution* μ on X is an additive map from the set of compact-opens in X to $\mathbb{Q}_p$ (the p-adic number filed); this means that if $U \subset X$ is the disjoint union of compact-open sets $U_1, U_2, \ldots, U_n$, then

$$\mu(U) = \mu(U_1) + \mu(U_2) + \cdots + \mu(U_n).$$

Now let X be a compact-open subset of $\mathbb{Q}_p$, such as $\mathbb{Z}_p$ (the ring of p-adic integers) or $\mathbb{Z}_p^\times$ (the group of p-adic units). A subset of type $a + (p^N) = a + p^N\mathbb{Z}_p$ is called an *interval*. The following proposition is a key result to prove the distribution property[1] p.32:

Proposition 1. *Every map μ from the set of intervals contained in X to $\mathbb{Q}_p$ for which*

$$(1) \qquad \mu(a + (p^N)) = \sum_{b=0}^{p-1} \mu(a + bp^N + (p^{N+1}))$$

whenever $a + (p^N) \subset X$, extends uniquely to a p-adic distribution on X.

Example: (i) The Haar distribution μ_{Haar}:

$$\mu_{\mathrm{Haar}}(a + (p^N)) := \frac{1}{p^N}.$$

(ii) The Dirac distribution μ_α ($\alpha \in \mathbb{Z}_p$):

$$\mu_\alpha(U) := \begin{cases} 1 & \text{if } \alpha \in U \\ 0 & \text{otherwise.} \end{cases}$$

(iii) The Mazur distribution μ_{Mazur}:
$$\mu_{\mathrm{Mazur}}(a + (p^N)) := \tfrac{a}{p^N} - \tfrac{1}{2}.$$

It is known that some of distributions are constructed by the Bernoulli polynomials. The k-th Bernoulli polynomial $B_k(x)$ is defined as

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.$$

We define a map $\mu_{B,k}$ on intervals $a + (p^N)$ by

$$\mu_{B,k}(a + (p^N)) = p^{N(k-1)} B_k\left(\frac{a}{p^N}\right).$$

It is known that the map satisfies the relation

$$\mu_{B,k}(a + (p^N)) = \sum_{b=0}^{p-1} \mu_{B,k}(a + bp^N + (p^{N+1})),$$

(see Koblitz's book[1] p.35). This shows that $\mu_{B,k}$ extends to a distribution on $\mathbb{Z}_p$ (called the "k-th Bernoulli distribution"). This construction is standard in a sense. In fact, the first few $B_k(x)$ give us the following distributions:

$$\mu_{B,0}(a + (p^N)) = p^{-N}, \quad \text{i.e.} \quad \mu_{B,0} = \mu_{\mathrm{Haar}},$$

$$\mu_{B,1}(a + (p^N)) = B_1\left(\tfrac{a}{p^N}\right) = \tfrac{a}{p^N} - \tfrac{1}{2},$$

i.e. $\mu_{B,1} = \mu_{\mathrm{Mazur}}$.

The main purpose of this note is to give an elementary proof of the following theorem[2].

Theorem 1. *Let $f_n(x)$ be a monic polynomial of degree n with rational coefficients. Let μ_{f_n} be the map on intervals $a + (p^N)$ defined by*

$$(2) \qquad \mu_{f_n}(a + (p^N)) = p^{N(n-1)} f_n\left(\frac{a}{p^N}\right).$$

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If μ_{f_n} satisfies the relation (1), then

$$f_n(x) = B_n(x).$$

This theorem means that the Bernoulli polynomials are the only polynomials that can be used to define distirbutions in this way.
To prove the theorem, we prepare the following lemma.

Lemma 1. *Let B_k $(k \geq 0)$ be the k-th Bernoulli number. Then, for any positive integer $i\,(i \geq 2)$ and N, we have*

$$(3) \quad \sum_{k=0}^{i-1} N^{k-i}\binom{i}{k}\cdot S_{i-k}(N-1) + (N - N^{1-i})B_i = 0,$$

where $S_k(m) := \sum_{a=1}^{m} a^k$.

Proof. The identity (3) is proved by the following identity on Bernoulli numbers:

$$(4) \qquad \sum_{k=0}^{i} N^{k-1}\binom{i}{k}B_k \cdot S_{i-k}(N) = B_i.$$

In fact, from this formula, we have

$$\sum_{k=0}^{i-1} N^{k-1}\binom{i}{k}B_k \cdot S_{i-k}(N) + N^i \cdot B_i = B_i, \ (i \geq 2).$$

Since $\sum_{k=0}^{i-1}\binom{i}{k}B_k = 0$, the above identity can be rewritten as

(5)

$$N^{i-1}\left(\sum_{k=0}^{i-1} N^{k-i}\binom{i}{k}B_k \cdot S_{i-k}(N-1)\right) + N^i \cdot B_i = B_i.$$

This proves (3).

Example: We shall give an example which shows the validity of (3).
The case that $i = 3$.

$$\sum_{k=0}^{2} N^{k-3}\binom{3}{k}B_k \cdot S_{3-k}(N-1) + (N - N^{-2})B_3$$
$$= \frac{1}{N^3}\left(\frac{N(N-1)}{2}\right)^2 - \frac{3}{2}\frac{1}{N^2}\frac{(N-1)N(2N-1)}{6} + \frac{1}{2}\frac{1}{N}\frac{(N-1)N}{2} = 0.$$

Special case of the identity (3) can be found in Yamaguchi's book[3].

Proof of Theorem : By assumption, $\mu_{f_n}(a + (p^N))$ satisfies the relation

$$\mu_{f_n}(a + (p^N)) = \sum_{b=0}^{p-1} \mu_{f_n}(a + bp^N + (p^{N+1}))$$

From this formula and the definition of μ_{f_n}, we have

$$(6) \qquad f_n(p\alpha) = p^{n-1}\sum_{b=0}^{p-1} f_n\left(\alpha + \frac{b}{p}\right).$$

where $\alpha = \frac{a}{p^{N+1}}$. Therefore, it suffices to show that if f_n satisfies the relation (6) for any $\alpha \in \mathbb{Q}_p$, then $f_n(x) = B_n(x)$. Now we write

$$f_n(x) = \sum_{i=0}^{n} a_i^{(n)} x^{n-i},$$

with $a_i^{(n)} \in \mathbb{Q}$ and $a_0^{(n)} = 1$. By the definition of the Bernoulli polynomial, our purpose is reduced to prove

$$(7) \qquad \frac{a_i^{(n)}}{\binom{n}{i}} = B_i, \quad (i = 1, \ldots, n).$$

To prove this, we compare the coefficients of α^{n-i} of the both sides of (6). This implies that

(8)

$$a_i^{(n)} p^{n-i} = p^{n-1}\left(\sum_{k=0}^{i-1} a_k^{(n)}\binom{n-k}{i-k}\frac{S_{i-k}(p-1)}{p^{i-k}} + p\,a_i^{(n)}\right).$$

We shall prove (7) by induction on i.
The case $i = 1$: Since $a_1^{(n)} = -\frac{n}{2}$, we have

$$\frac{a_1^{(n)}}{\binom{n}{1}} = \frac{-\frac{n}{2}}{n} = -\frac{1}{2} = B_1.$$

This shows that the identity (7) is true for $i = 1$.
Next we assume that

$$\frac{a_1^{(n)}}{\binom{n}{1}} = B_1, \frac{a_2^{(n)}}{\binom{n}{2}} = B_2, \ldots, \frac{a_{i-1}^{(n)}}{\binom{n}{i-1}} = B_{i-1}.$$

Substituting these formulas for the right hand side of (8) and using well-known properties of the binomial coefficients, we have

(9)

$$a_i^{(n)}p^{n-i} = p^{n-1}\left(\binom{n}{i}\sum_{k=0}^{i-1}\binom{i}{k}B_k\frac{S_{i-k}(p-1)}{p^{i-k}} + pa_i^{(n)}\right).$$

Therefore, we get

$$\sum_{k=0}^{i-1}p^{k-i}\binom{i}{k}B_kS_{i-k}(p-1) + (p-p^{1-i})\frac{a_i^{(n)}}{\binom{n}{i}} = 0.$$

Comparing with this identity and (3) for $N = p$, we have

$$\frac{a_i^{(n)}}{\binom{n}{i}} = B_i.$$

This completes the inductive argument.

References

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