均衡型 C_k-Foil デザイン

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Balanced C_k-Foil Designs

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In graph theory, the decomposition problem of graphs is a very important topic. Various type of decompositions of many graphs can be seen in the literature of gaph theory.

Let k , t and n be positive integers ($k \geq 3$, $t \geq 2$). It is shown that the necessary and sufficient condition for the existence of a balanced C_k-t-foil decomposition of the complete graph K_n is $n \equiv 1 \pmod{2kt}$ for $k = 11, 13, 15, 17, 19, 21, 23, 25$. These decompositions are to be known as *balanced C_k-foil designs.*

Key words: Balanced C_k-foil decomposition, Complete graph, Graph theory

1. Introduction

Let k ($k \geq 3$), t ($t \geq 2$) and n be positive integers. Let K_n denote *the complete graph* of n vertices. Let C_k be *the k-cycle* (or *the cycle* on k vertices). *The C_k-t-foil* (or *the C_k-t-windmill*) is a graph of t edge-disjoint C_k's with a common vertex and the common vertex is called *the center of the C_k-t-foil.* In particular, the C_k-2-foil and the C_k-3-foil are called *the C_k-bowtie* and *the C_k-trefoil*, respectively. When K_n is decomposed into edge-disjoint sum of C_k-t-foils, it is called that *K_n has a C_k-t-foil decomposition.* Moreover, when every vertex of K_n appears in the same number of C_k-t-foils, it is called that *K_n has a balanced C_k-t-foil decomposition* and this number is called *the replication number.*
We show that the necessary and sufficient condition for the existence of a balanced C_k-t-foil decomposition of the complete graph K_n is $n \equiv 1 \pmod{2kt}$ for $k = 11, 13, 15, 17, 19, 21, 23, 25$.

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or $3 \pmod 6$. This decomposition is known as *a Steiner triple system.* See Colbourn and Rosa [1], Rosa [2], and Wallis [3].

Horák and Rosa [4] proved that K_n has a C_3-bowtie decomposition if and only if $n \equiv 1$ or $9 \pmod{12}$. This decomposition is known as *a bowtie system.* Our balanced C_k-t-foil decomposition of K_n is to be known as *a balanced C_k-foil design.*
For the design theory, see Colbourn [5], Lindner [6], and Ushio [7]. For the graph decomposition, see Ushio [8, 9], Ushio and Fujimoto [10 – 16].

2. Balanced C_k-foil designs

Notation. We consider the vertex set V of K_n as $V = \{1, 2, ..., n\}$.

Theorem 1. If K_n has a balanced C_k-t-foil decomposition, then $n \equiv 1 \pmod{2kt}$.

Proof. Suppose that K_n has a balanced C_k-t-foil decomposition. Let b be the number of C_k-t-foils and r be the replication number. Then $b = n(n-1)/2kt$ and $r = ((k-1)t+1)(n-1)/2kt$. Among r C_k-t-foils having a vertex v of K_n, let r_1 and r_2 be the numbers of C_k-t-foils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $2tr_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/2kt$ and $r_2 = (k-1)(n-1)/2k$. Therefore, $n \equiv 1 \pmod{2kt}$ is necessary.

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In the following sections 3–10, we will prove the following theorem.

Theorem 2. K_n has a balanced C_k-t-foil decomposition if and only if $n \equiv 1 \pmod{2kt}$ for $k = 11, 13, 15, 17, 19, 21, 23, 25$.

3. Balanced C_{11}-foil designs

Theorem 3. K_n has a balanced C_{11}-t-foil decomposition if and only if $n \equiv 1 \pmod{22t}$.

Proof. (Necessity) Obvious by Theorem 1.
(Sufficiency) Put $n = 22st + 1, T = st$. Then $n = 22T + 1$. Construct a C_{11}-T-foil as follows:
$\{ (22T+1, T, 15T, 21T+1, 9T+1, 10T+2, T+1,$
$4T+2, 12T+2, 6T+2, 2T+1),$
$(22T+1, T-1, 15T-2, 21T, 9T+2, 10T+4, T+2,$
$4T+4, 12T+3, 6T+4, 2T+2),$
$(22T+1, T-2, 15T-4, 21T-1, 9T+3, 10T+6,$
$T+3, 4T+6, 12T+4, 6T+6, 2T+3),$
...,
$(22T+1, 1, 13T+2, 20T+2, 10T, 12T, 2T, 6T,$
$13T+1, 8T, 3T) \}.$
Decompose this C_{11}-T-foil into s C_{11}-t-foils. Then these starters comprise a balanced C_{11}-t-foil decomposition of K_n.

Example 3.1. Balanced C_{11}-decomposition of K_{23}.
$\{(23, 1, 15, 22, 10, 12, 2, 6, 14, 8, 3)\}$.
This stater comprises a balanced C_{11}-decomposition of K_{23}.

Example 3.2. Balanced C_{11}-2-foil decomposition of K_{45}.
$\{(45, 2, 30, 43, 19, 22, 3, 10, 26, 14, 5),$
$(45, 1, 28, 42, 20, 24, 4, 12, 27, 16, 6)\}$.
This stater comprises a balanced C_{11}-2-foil decomposition of K_{45}.

Example 3.3. Balanced C_{11}-3-foil decomposition of K_{67}.
$\{(67, 3, 45, 64, 28, 32, 4, 14, 38, 20, 7),$
$(67, 2, 43, 63, 29, 34, 5, 16, 39, 22, 8),$
$(67, 1, 41, 62, 30, 36, 6, 18, 40, 24, 9)\}$.
This stater comprises a balanced C_{11}-3-foil decomposition of K_{67}.

Example 3.4. Balanced C_{11}-4-foil decomposition of K_{89}.
$\{(89, 4, 60, 85, 37, 42, 5, 18, 50, 26, 9),$
$(89, 3, 58, 84, 38, 44, 6, 20, 51, 28, 10),$
$(89, 2, 56, 83, 39, 46, 7, 22, 52, 30, 11),$
$(89, 1, 54, 82, 40, 48, 8, 24, 53, 32, 12)\}$.
This stater comprises a balanced C_{11}-4-foil decomposition of K_{89}.

Example 3.5. Balanced C_{11}-5-foil decomposition of K_{111}.
$\{(111, 5, 75, 106, 46, 52, 6, 22, 62, 32, 11),$
$(111, 4, 73, 105, 47, 54, 7, 24, 63, 34, 12),$
$(111, 3, 71, 104, 48, 56, 8, 26, 64, 36, 13),$
$(111, 2, 69, 103, 49, 58, 9, 28, 65, 38, 14),$
$(111, 1, 67, 102, 50, 60, 10, 30, 66, 40, 15)\}$.
This stater comprises a balanced C_{11}-5-foil decomposition of K_{111}.

Example 3.6. Balanced C_{11}-6-foil decomposition of K_{133}.
$\{(133, 6, 90, 127, 55, 62, 7, 26, 74, 38, 13),$
$(133, 5, 88, 126, 56, 64, 8, 28, 75, 40, 14),$
$(133, 4, 86, 125, 57, 66, 9, 30, 76, 42, 15),$
$(133, 3, 84, 124, 58, 68, 10, 32, 77, 44, 16),$
$(133, 2, 82, 123, 59, 70, 11, 34, 78, 46, 17),$
$(133, 1, 80, 122, 60, 72, 12, 36, 79, 48, 18)\}$.
This stater comprises a balanced C_{11}-6-foil decomposition of K_{133}.

Corollary 3.1. K_n has a balanced C_{11}-bowtie decomposition if and only if $n \equiv 1 \pmod{44}$.

Corollary 3.2. K_n has a balanced C_{11}-trefoil decomposition if and only if $n \equiv 1 \pmod{66}$.

4. Balanced C_{13}-foil designs

Theorem 4. K_n has a balanced C_{13}-t-foil decomposition if and only if $n \equiv 1 \pmod{26t}$.

Proof. (Necessity) Obvious by Theorem 1.
(Sufficiency) Put $n = 26st + 1, T = st$. Then $n = 26T + 1$. Construct a C_{13}-T-foil as follows:
$\{ (26T+1, T, 12T, 23T+1, 14T, 15T+1, T+1,$
$19T+2, 24T+2, 3T+2, 23T+2, 5T+2, 2T+1),$
$(26T+1, T-1, 12T-2, 23T, 14T-2, 15T, T+2,$
$19T+4, 24T+3, 3T+4, 23T+3, 5T+4, 2T+2),$
$(26T+1, T-2, 12T-4, 23T-1, 14T-4, 15T-1,$
$T+3, 19T+6, 24T+4, 3T+6, 23T+4, 5T+6, 2T+$
$3),$
...,
$(26T+1, 1, 10T+2, 22T+2, 12T+2, 14T+2, 2T,$

$21T, 25T+1, 5T, 24T+1, 7T, 3T)$ }.
Decompose this C_{13}-T-foil into s C_{13}-t-foils. Then these starters comprise a balanced C_{13}-t-foil decomposition of K_n.

Example 4.1. Balanced C_{13}-decomposition of K_{27}.
$\{(27,1,12,24,14,16,2,21,26,5,25,7,3)\}$.
This stater comprises a balanced C_{13}-decomposition of K_{27}.

Example 4.2. Balanced C_{13}-2-foil decomposition of K_{53}.
$\{(53,2,24,47,28,31,3,40,50,8,48,12,5),$
$(53,1,22,46,26,30,4,42,51,10,49,14,6)\}$.
This stater comprises a balanced C_{13}-2-foil decomposition of K_{53}.

Example 4.3. Balanced C_{13}-3-foil decomposition of K_{79}.
$\{(79,3,36,70,42,46,4,59,74,11,71,17,7),$
$(79,2,34,69,40,45,5,61,75,13,72,19,8),$
$(79,1,32,68,38,44,6,63,76,15,73,21,9)\}$.
This stater comprises a balanced C_{13}-3-foil decomposition of K_{79}.

Example 4.4. Balanced C_{13}-4-foil decomposition of K_{105}.
$\{(105,4,48,93,56,61,5,78,98,14,94,22,9),$
$(105,3,46,92,54,60,6,80,99,16,95,24,10),$
$(105,2,44,91,52,59,7,82,100,18,96,26,11),$
$(105,1,42,90,50,58,8,84,101,20,97,28,12)\}$.
This stater comprises a balanced C_{13}-4-foil decomposition of K_{105}.

Example 4.5. Balanced C_{13}-5-foil decomposition of K_{131}.
$\{(131,5,60,116,70,76,6,97,122,17,117,27,11),$
$(131,4,58,115,68,75,7,99,123,19,118,29,12),$
$(131,3,56,114,66,74,8,101,124,21,119,31,13),$
$(131,2,54,113,64,73,9,103,125,23,120,33,14),$
$(131,1,52,112,62,72,10,105,126,25,121,35,15)\}$.
This stater comprises a balanced C_{13}-5-foil decomposition of K_{131}.

Example 4.6. Balanced C_{13}-6-foil decomposition of K_{157}.
$\{(157,6,72,139,84,91,7,116,146,20,140,32,13),$
$(157,5,70,138,82,90,8,118,147,22,141,34,14),$
$(157,4,68,137,80,89,9,120,148,24,142,36,15),$
$(157,3,66,136,78,88,10,122,149,26,143,38,16),$
$(157,2,64,135,76,87,11,124,150,28,144,40,17),$
$(157,1,62,134,74,86,12,126,151,30,145,42,18)\}$.
This stater comprises a balanced C_{13}-6-foil decomposition of K_{157}.

Corollary 4.1. K_n has a balanced C_{13}-bowtie decomposition if and only if $n \equiv 1 \pmod{52}$.

Corollary 4.2. K_n has a balanced C_{13}-trefoil decomposition if and only if $n \equiv 1 \pmod{78}$.

5. Balanced C_{15}-foil designs

Theorem 5. K_n has a balanced C_{15}-t-foil decomposition if and only if $n \equiv 1 \pmod{30t}$.

Proof. (Necessity) Obvious by Theorem 1.
(Sufficiency) Put $n = 30st+1, T = st$. Then $n = 30T+1$. Construct a C_{15}-T-foil as follows:
$\{ (30T+1, T, 18T, 28T+1, 12T+1, 20T+2,$
$8T+1, 10T+2, 14T+2, 20T+3, 9T+2, 18T+3,$
$13T+2, 5T+2, T+1),$
$(30T+1, T-1, 18T-2, 28T, 12T+2, 20T+4,$
$8T+2, 10T+4, 14T+3, 20T+5, 9T+3, 18T+5,$
$13T+3, 5T+4, T+2),$
$(30T+1, T-2, 18T-4, 28T-1, 12T+3, 20T+6,$
$8T+3, 10T+6, 14T+4, 20T+7, 9T+4, 18T+7,$
$13T+4, 5T+6, T+3),$
...,
$(30T+1, 1, 16T+2, 27T+2, 13T, 22T, 9T, 12T,$
$15T+1, 22T+1, 10T+1, 20T+1, 14T+1, 7T, 2T)$
$\}$.
Decompose this C_{15}-T-foil into s C_{15}-t-foils. Then these starters comprise a balanced C_{15}-t-foil decomposition of K_n.

Example 5.1. Balanced C_{15}-decomposition of K_{31}.
$\{(31,1,18,29,13,22,9,12,16,23,11,21,15,7,2)\}$.
This stater comprises a balanced C_{15}-decomposition of K_{31}.

Example 5.2. Balanced C_{15}-2-foil decomposition of K_{61}.
$\{(61,2,36,57,25,42,17,22,30,43,20,39,28,12,3),$
$(61,1,34,56,26,44,18,24,31,45,21,41,29,14,4)\}$.
This stater comprises a balanced C_{15}-2-foil decomposition of K_{61}.

Example 5.3. Balanced C_{15}-3-foil decomposition of K_{91}.
$\{(91,3,54,85,37,62,25,32,44,63,29,57,41,17,4),$

(91, 2, 52, 84, 38, 64, 26, 34, 45, 65, 30, 59, 42, 19, 5),
(91, 1, 50, 83, 39, 66, 27, 36, 46, 67, 31, 61, 43, 21, 6)}.
This stater comprises a balanced C_{15}-3-foil decomposition of K_{91}.

Example 5.4. Balanced C_{15}-4-foil decomposition of K_{121}.
{(121, 4, 72, 113, 49, 82, 33, 42, 58, 83, 38, 75, 54, 22, 5),
(121, 3, 70, 112, 50, 84, 34, 44, 59, 85, 39, 77, 55, 24, 6),
(121, 2, 68, 111, 51, 86, 35, 46, 60, 87, 40, 79, 56, 26, 7),
(121, 1, 66, 110, 52, 88, 36, 48, 61, 89, 41, 81, 57, 28, 8)}.
This stater comprises a balanced C_{15}-4-foil decomposition of K_{121}.

Example 5.5. Balanced C_{15}-5-foil decomposition of K_{151}.
{(151, 5, 90, 141, 61, 102, 41, 52, 72, 103, 47, 93, 67, 27, 6),
(151, 4, 88, 140, 62, 104, 42, 54, 73, 105, 48, 95, 68, 29, 7),
(151, 3, 86, 139, 63, 106, 43, 56, 74, 107, 49, 97, 69, 31, 8),
(151, 2, 84, 138, 64, 108, 44, 58, 75, 109, 50, 99, 70, 33, 9),
(151, 1, 82, 137, 65, 110, 45, 60, 76, 111, 51, 101, 71, 35, 10)}.
This stater comprises a balanced C_{15}-5-foil decomposition of K_{151}.

Example 5.6. Balanced C_{15}-6-foil decomposition of K_{181}.
{(181, 6, 108, 169, 73, 122, 49, 62, 86, 123, 56, 111, 80, 32, 7),
(181, 5, 106, 168, 74, 124, 50, 64, 87, 125, 57, 113, 81, 34, 8),
(181, 4, 104, 167, 75, 126, 51, 66, 88, 127, 58, 115, 82, 36, 9),
(181, 3, 102, 166, 76, 128, 52, 68, 89, 129, 59, 117, 83, 38, 10),
(181, 2, 100, 165, 77, 130, 53, 70, 90, 131, 60, 119, 84, 40, 11),
(181, 1, 98, 164, 78, 132, 54, 72, 91, 133, 61, 121, 85, 42, 12)}.
This stater comprises a balanced C_{15}-6-foil decomposition of K_{181}.

Corollary 5.1. K_n has a balanced C_{15}-bowtie decomposition if and only if $n \equiv 1 \pmod{60}$.

Corollary 5.2. K_n has a balanced C_{15}-trefoil decomposition if and only if $n \equiv 1 \pmod{90}$.

6. Balanced C_{17}-foil designs

Theorem 6. K_n has a balanced C_{17}-t-foil decomposition if and only if $n \equiv 1 \pmod{34t}$.

Proof. (Necessity) Obvious by Theorem 1.
(Sufficiency) Put $n = 34st + 1, T = st$. Then $n = 34T + 1$. Construct a C_{17}-T-foil as follows:
{ $(34T + 1, T, 16T, 31T + 1, 13T + 1, 17T + 2, 4T+1, 6T+2, 16T+2, 23T+3, 11T+2, 17T+3, 29T + 3, 20T + 3, 19T + 2, 8T + 2, 3T + 1)$,
$(34T + 1, T - 1, 16T - 2, 31T, 13T + 2, 17T + 4, 4T+2, 6T+4, 16T+3, 23T+5, 11T+3, 17T+5, 29T + 4, 20T + 5, 19T + 3, 8T + 4, 3T + 2)$,
$(34T+1, T-2, 16T-4, 31T-1, 13T+3, 17T+6, 4T+3, 6T+6, 16T+4, 23T+7, 11T+4, 17T+7, 29T + 5, 20T + 7, 19T + 4, 8T + 6, 3T + 3)$,
...,
$(34T + 1, 2, 14T + 4, 30T + 3, 14T - 1, 19T - 2, 5T-1, 8T-2, 17T, 25T-1, 12T, 19T-1, 30T+1, 22T - 1, 20T, 10T - 2, 4T - 1)$,
$(34T + 1, 1, 14T + 2, 30T + 2, 14T, 19T, 5T, 8T, 17T + 1, 25T + 1, 12T + 1, 19T + 1, 11T, 22T + 1, 20T + 1, 10T, 4T)$ }.
Decompose this C_{17}-T-foil into s C_{17}-t-foils. Then these starters comprise a balanced C_{17}-t-foil decomposition of K_n.

Example 6.1. Balanced C_{17}-decomposition of K_{35}.
{(35, 1, 16, 32, 14, 19, 5, 8, 18, 26, 13, 20, 11, 23, 21, 10, 4)}.
This stater comprises a balanced C_{17}-decomposition of K_{35}.

Example 6.2. Balanced C_{17}-2-foil decomposition of K_{69}.
{(69, 2, 32, 63, 27, 36, 9, 14, 34, 49, 24, 37, 61, 43, 40, 18, 7),
(69, 1, 30, 62, 28, 38, 10, 16, 35, 51, 25, 39, 22, 45, 41, 20, 8)}.
This stater comprises a balanced C_{17}-2-foil decomposition of K_{69}.

Example 6.3. Balanced C_{17}-3-foil decomposition of K_{103}.
{(103, 3, 48, 94, 40, 53, 13, 20, 50, 72, 35, 54, 90, 63, 59,

26, 10),
(103, 2, 46, 93, 41, 55, 14, 22, 51, 74, 36, 56, 91, 65, 60, 28, 11),
(103, 1, 44, 92, 42, 57, 15, 24, 52, 76, 37, 58, 33, 67, 61, 30, 12)}.
This stater comprises a balanced C_{17}-3-foil decomposition of K_{103}.

Example 6.4. Balanced C_{17}-4-foil decomposition of K_{137}.
{(137, 4, 64, 125, 53, 70, 17, 26, 66, 95, 46, 71, 119, 83, 78, 34, 13),
(137, 3, 62, 124, 54, 72, 18, 28, 67, 97, 47, 73, 120, 85, 79, 36, 14),
(137, 2, 60, 123, 55, 74, 19, 30, 68, 99, 48, 75, 121, 87, 80, 38, 15),
(137, 1, 58, 122, 56, 76, 20, 32, 69, 101, 49, 77, 44, 89, 81, 40, 16)}.
This stater comprises a balanced C_{17}-4-foil decomposition of K_{137}.

Example 6.5. Balanced C_{17}-5-foil decomposition of K_{171}.
{(171, 5, 80, 156, 66, 87, 21, 32, 82, 118, 57, 88, 148, 103, 97, 42, 16),
(171, 4, 78, 155, 67, 89, 22, 34, 83, 120, 58, 90, 149, 105, 98, 44, 17),
(171, 3, 76, 154, 68, 91, 23, 36, 84, 122, 59, 92, 150, 107, 99, 46, 18),
(171, 2, 74, 153, 69, 93, 24, 38, 85, 124, 60, 94, 151, 109, 100, 48, 19),
(171, 1, 72, 152, 70, 95, 25, 40, 86, 126, 61, 96, 55, 111, 101, 50, 20)}.
This stater comprises a balanced C_{17}-5-foil decomposition of K_{171}.

Example 6.6. Balanced C_{17}-6-foil decomposition of K_{205}.
{(205, 6, 96, 187, 79, 104, 25, 38, 98, 141, 68, 105, 177, 123, 116, 50, 19),
(205, 5, 94, 186, 80, 106, 26, 40, 99, 143, 69, 107, 178, 125, 117, 52, 20),
(205, 4, 92, 185, 81, 108, 27, 42, 100, 145, 70, 109, 179, 127, 118, 54, 21),
(205, 3, 90, 184, 82, 110, 28, 44, 101, 147, 71, 111, 180, 129, 119, 56, 22),
(205, 2, 88, 183, 83, 112, 29, 46, 102, 149, 72, 113, 181, 131, 120, 58, 23),
(205, 1, 86, 182, 84, 114, 30, 48, 103, 151, 73, 115, 66, 133, 121, 60, 24)}.
This stater comprises a balanced C_{17}-6-foil decomposition of K_{205}.

Corollary 6.1. K_n has a balanced C_{17}-bowtie decomposition if and only if $n \equiv 1 \pmod{68}$.

Corollary 6.2. K_n has a balanced C_{17}-trefoil decomposition if and only if $n \equiv 1 \pmod{102}$.

7. Balanced C_{19}-foil designs

Theorem 7. K_n has a balanced C_{19}-t-foil decomposition if and only if $n \equiv 1 \pmod{38t}$.

Proof. **(Necessity)** Obvious by Theorem 1.
(Sufficiency) Put $n = 38st + 1, T = st$. Then $n = 38T + 1$.
Case 1. n = 39. (Example 7.1. Balanced C_{19}-decomposition of K_{39}.)
Case 2. n = 38T + 1, T ≥ 2. Construct a C_{19}-T-foil as follows:
{ $(38T+1, T, 16T, 36T+1, 16T+1, 26T+2, 10T+1, 11T+2, 17T+2, 21T+3, 29T+3, 6T+3, 18T+3, 14T+3, 5T+2, 30T+3, 24T+2, 21T+2, 13T+1)$,
$(38T+1, T-1, 16T-2, 36T, 16T+2, 26T+4, 10T+2, 11T+4, 17T+3, 21T+5, 29T+4, 6T+5, 18T+4, 14T+5, 5T+3, 30T+5, 24T+3, 21T+4, 13T+2)$,
$(38T+1, T-2, 16T-4, 36T-1, 16T+3, 26T+6, 10T+3, 11T+6, 17T+4, 21T+7, 29T+5, 6T+7, 18T+5, 14T+7, 5T+4, 30T+7, 24T+4, 21T+6, 13T+3)$,
...,
$(38T+1, 2, 14T+4, 35T+3, 17T-1, 28T-2, 11T-1, 13T-2, 18T, 23T-1, 30T+1, 8T-1, 19T+1, 16T-1, 6T, 32T-1, 25T, 23T-2, 14T-1)$,
$(38T+1, 1, 14T+2, 35T+2, 17T, 28T, 11T, 13T, 18T+1, 23T+1, 30T+2, 8T+1, 19T+2, 9T+2, 6T+1, 32T+1, 25T+1, 23T, 14T)$ }.
Decompose this C_{19}-T-foil into s C_{19}-t-foils. Then these starters comprise a balanced C_{19}-t-foil decomposition of K_n.

Example 7.1. Balanced C_{19}-decomposition of K_{39}.
{(39, 1, 16, 37, 17, 19, 2, 13, 18, 24, 32, 9, 21, 11, 7, 33, 26, 23, 14)}.
This stater comprises a balanced C_{19}-decomposition of K_{39}.

Example 7.2. Balanced C_{19}-2-foil decomposition of K_{77}.

{(77, 2, 32, 73, 33, 54, 21, 24, 36, 45, 61, 15, 39, 31, 12, 63, 50, 44, 27),
(77, 1, 30, 72, 34, 56, 22, 26, 37, 47, 62, 17, 40, 20, 13, 65, 51, 46, 28)}.
This stater comprises a balanced C_{19}-2-foil decomposition of K_{77}.

Example 7.3. Balanced C_{19}-3-foil decomposition of K_{115}.
{(115, 3, 48, 109, 49, 80, 31, 35, 53, 66, 90, 21, 57, 45, 17, 93, 74, 65, 40),
(115, 2, 46, 108, 50, 82, 32, 37, 54, 68, 91, 23, 58, 47, 18, 95, 75, 67, 41),
(115, 1, 44, 107, 51, 84, 33, 39, 55, 70, 92, 25, 59, 29, 19, 97, 76, 69, 42)}.
This stater comprises a balanced C_{19}-3-foil decomposition of K_{115}.

Example 7.4. Balanced C_{19}-4-foil decomposition of K_{153}.
{(153, 4, 64, 145, 65, 106, 41, 46, 70, 87, 119, 27, 75, 59, 22, 123, 98, 86, 53),
(153, 3, 62, 144, 66, 108, 42, 48, 71, 89, 120, 29, 76, 61, 23, 125, 99, 88, 54),
(153, 2, 60, 143, 67, 110, 43, 50, 72, 91, 121, 31, 77, 63, 24, 127, 100, 90, 55),
(153, 1, 58, 142, 68, 112, 44, 52, 73, 93, 122, 33, 78, 38, 25, 129, 101, 92, 56)}.
This stater comprises a balanced C_{19}-4-foil decomposition of K_{153}.

Example 7.5. Balanced C_{19}-5-foil decomposition of K_{191}.
{(191, 5, 80, 181, 81, 132, 51, 57, 87, 108, 148, 33, 93, 73, 27, 153, 122, 107, 66),
(191, 4, 78, 180, 82, 134, 52, 59, 88, 110, 149, 35, 94, 75, 28, 155, 123, 109, 67),
(191, 3, 76, 179, 83, 136, 53, 61, 89, 112, 150, 37, 95, 77, 29, 157, 124, 111, 68),
(191, 2, 74, 178, 84, 138, 54, 63, 90, 114, 151, 39, 96, 79, 30, 159, 125, 113, 69),
(191, 1, 72, 177, 85, 140, 55, 65, 91, 116, 152, 41, 97, 47, 31, 161, 126, 115, 70)}.
This stater comprises a balanced C_{19}-5-foil decomposition of K_{191}.

Example 7.6. Balanced C_{19}-6-foil decomposition of K_{229}.
{(229, 6, 96, 217, 97, 158, 61, 68, 104, 129, 177, 39, 111, 87, 32, 183, 146, 128, 79),
(229, 5, 94, 216, 98, 160, 62, 70, 105, 131, 178, 41, 112, 89, 33, 185, 147, 130, 80),
(229, 4, 92, 215, 99, 162, 63, 72, 106, 133, 179, 43, 113, 91, 34, 187, 148, 132, 81),
(229, 3, 90, 214, 100, 164, 64, 74, 107, 135, 180, 45, 114, 93, 35, 189, 149, 134, 82),
(229, 2, 88, 213, 101, 166, 65, 76, 108, 137, 181, 47, 115, 95, 36, 191, 150, 136, 83),
(229, 1, 86, 212, 102, 168, 66, 78, 109, 139, 182, 49, 116, 56, 37, 193, 151, 138, 84)}.
This stater comprises a balanced C_{19}-6-foil decomposition of K_{229}.

Corollary 7.1. K_n has a balanced C_{19}-bowtie decomposition if and only if $n \equiv 1 \pmod{76}$.

Corollary 7.2. K_n has a balanced C_{19}-trefoil decomposition if and only if $n \equiv 1 \pmod{114}$.

8. Balanced C_{21}-foil designs

Theorem 8. K_n has a balanced C_{21}-t-foil decomposition if and only if $n \equiv 1 \pmod{42t}$.

Proof. (Necessity) Obvious by Theorem 1.
(Sufficiency) Put $n = 42st + 1, T = st$. Then $n = 42T + 1$. Construct a C_{21}-T-foil as follows:
{ $(42T + 1, T, 20T, 39T + 1, 17T + 1, 20T + 2, 3T + 1, 4T + 2, 15T + 2, 22T + 3, 7T + 2, 13T + 3, 25T+3, 11T+3, 26T+3, 20T+3, 16T+2, 32T+3, 24T + 2, 11T + 2, 2T + 1)$,
$(42T + 1, T - 1, 20T - 2, 39T, 17T + 2, 20T + 4, 3T + 2, 4T + 4, 15T + 3, 22T + 5, 7T + 3, 13T + 5, 25T+4, 11T+5, 26T+4, 20T+5, 16T+3, 32T+5, 24T + 3, 11T + 4, 2T + 2)$,
$(42T+1, T-2, 20T-4, 39T-1, 17T+3, 20T+6, 3T + 3, 4T + 6, 15T + 4, 22T + 7, 7T + 4, 13T + 7, 25T+5, 11T+7, 26T+5, 20T+7, 16T+4, 32T+7, 24T + 4, 11T + 6, 2T + 3)$,
...,
$(42T + 1, 2, 18T + 4, 38T + 3, 18T - 1, 22T - 2, 4T-1, 6T-2, 16T, 24T-1, 8T, 15T-1, 26T+1, 13T - 1, 27T + 1, 22T - 1, 17T, 34T - 1, 25T, 13T - 2, 3T - 1)$,
$(42T + 1, 1, 18T + 2, 38T + 2, 18T, 22T, 4T, 6T, 16T+1, 24T+1, 8T+1, 15T+1, 26T+2, 13T+1, 27T + 2, 22T + 2, 39T + 2, 34T + 1, 25T + 1, 13T, 3T)$ }.
Decompose this C_{21}-T-foil into s C_{21}-t-foils. Then these starters comprise a balanced C_{21}-t-foil decomposition of K_n.

Example 8.1. Balanced C_{21}-decomposition of K_{43}.

{(43, 1, 20, 40, 18, 22, 4, 6, 17, 25, 9, 16, 28, 14, 29, 24, 41, 35, 26, 13, 3)}.
This stater comprises a balanced C_{21}-decomposition of K_{43}.

Example 8.2. Balanced C_{21}-2-foil decomposition of K_{85}.
{(85, 2, 40, 79, 35, 42, 7, 10, 32, 47, 16, 29, 53, 25, 55, 43, 34, 67, 50, 24, 5),
(85, 1, 38, 78, 36, 44, 8, 12, 33, 49, 17, 31, 54, 27, 56, 46, 80, 69, 51, 26, 6)}.
This stater comprises a balanced C_{21}-2-foil decomposition of K_{85}.

Example 8.3. Balanced C_{21}-3-foil decomposition of K_{127}.
{(127, 3, 60, 118, 52, 62, 10, 14, 47, 69, 23, 42, 78, 36, 81, 63, 50, 99, 74, 35, 7),
(127, 2, 58, 117, 53, 64, 11, 16, 48, 71, 24, 44, 79, 38, 82, 65, 51, 101, 75, 37, 8),
(127, 1, 56, 116, 54, 66, 12, 18, 49, 73, 25, 46, 80, 40, 83, 68, 119, 103, 76, 39, 9)}.
This stater comprises a balanced C_{21}-3-foil decomposition of K_{127}.

Example 8.4. Balanced C_{21}-4-foil decomposition of K_{169}.
{(169, 4, 80, 157, 69, 82, 13, 18, 62, 91, 30, 55, 103, 47, 107, 83, 66, 131, 98, 46, 9),
(169, 3, 78, 156, 70, 84, 14, 20, 63, 93, 31, 57, 104, 49, 108, 85, 67, 133, 99, 48, 10),
(169, 2, 76, 155, 71, 86, 15, 22, 64, 95, 32, 59, 105, 51, 109, 87, 68, 135, 100, 50, 11),
(169, 1, 74, 154, 72, 88, 16, 24, 65, 97, 33, 61, 106, 53, 110, 90, 158, 137, 101, 52, 12)}.
This stater comprises a balanced C_{21}-4-foil decomposition of K_{169}.

Example 8.5. Balanced C_{21}-5-foil decomposition of K_{211}.
{(211, 5, 100, 196, 86, 102, 16, 22, 77, 113, 37, 68, 128, 58, 133, 103, 82, 163, 122, 57, 11),
(211, 4, 98, 195, 87, 104, 17, 24, 78, 115, 38, 70, 129, 60, 134, 105, 83, 165, 123, 59, 12),
(211, 3, 96, 194, 88, 106, 18, 26, 79, 117, 39, 72, 130, 62, 135, 107, 84, 167, 124, 61, 13),
(211, 2, 94, 193, 89, 108, 19, 28, 80, 119, 40, 74, 131, 64, 136, 109, 85, 169, 125, 63, 14),
(211, 1, 92, 192, 90, 110, 20, 30, 81, 121, 41, 76, 132, 66, 137, 112, 197, 171, 126, 65, 15)}.
This stater comprises a balanced C_{21}-5-foil decomposition of K_{211}.

Example 8.6. Balanced C_{21}-6-foil decomposition of K_{253}.
{(253, 6, 120, 235, 103, 122, 19, 26, 92, 135, 44, 81, 153, 69, 159, 123, 98, 195, 146, 68, 13),
(253, 5, 118, 234, 104, 124, 20, 28, 93, 137, 45, 83, 154, 71, 160, 125, 99, 197, 147, 70, 14),
(253, 4, 116, 233, 105, 126, 21, 30, 94, 139, 46, 85, 155, 73, 161, 127, 100, 199, 148, 72, 15),
(253, 3, 114, 232, 106, 128, 22, 32, 95, 141, 47, 87, 156, 75, 162, 129, 101, 201, 149, 74, 16),
(253, 2, 112, 231, 107, 130, 23, 34, 96, 143, 48, 89, 157, 77, 163, 131, 102, 203, 150, 76, 17),
(253, 1, 110, 230, 108, 132, 24, 36, 97, 145, 49, 91, 158, 79, 164, 134, 236, 205, 151, 78, 18)}.
This stater comprises a balanced C_{21}-6-foil decomposition of K_{253}.

Corollary 8.1. K_n has a balanced C_{21}-bowtie decomposition if and only if $n \equiv 1 \pmod{84}$.

Corollary 8.2. K_n has a balanced C_{21}-trefoil decomposition if and only if $n \equiv 1 \pmod{126}$.

9. Balanced C_{23}-foil designs

Theorem 9. K_n has a balanced C_{23}-t-foil decomposition if and only if $n \equiv 1 \pmod{46t}$.

Proof. **(Necessity)** Obvious by Theorem 1.
(Sufficiency) Put $n = 46st + 1, T = st$. Then $n = 46T + 1$.
Case 1. n = 47. (Example 9.1. Balanced C_{23}-decomposition of K_{47}.)
Case 2. n = 46T + 1, T ≥ 2. Construct a C_{23}-T-foil as follows:
{ $(46T+1, T, 20T, 44T+1, 20T+1, 24T+2, 4T+1, 16T+2, 26T+2, 32T+2, 34T+3, 14T+3, 28T+3, 36T+4, 21T+3, 38T+4, 9T+3, 40T+4, 29T+3, 18T+3, 11T+2, 7T+2, 6T+1)$,
$(46T+1, T-1, 20T-2, 44T, 20T+2, 24T+4, 4T+2, 16T+4, 22T+3, 24T+5, 34T+4, 14T+5, 28T+4, 36T+6, 21T+4, 38T+6, 9T+4, 40T+6, 29T+4, 18T+5, 11T+3, 7T+4, 6T+2)$,
$(46T+1, T-2, 20T-4, 44T-1, 20T+3, 24T+6, 4T+3, 16T+6, 22T+4, 24T+7, 34T+5, 14T+7, 28T+5, 36T+8, 21T+5, 38T+8, 9T+5, 40T+8, 29T+5, 18T+7, 11T+4, 7T+6, 6T+3)$,
$(46T+1, T-3, 20T-6, 44T-2, 20T+4, 24T+8, 4T+4, 16T+8, 22T+5, 24T+9, 34T+6, 14T+9, 28T+6, 36T+10, 21T+6, 38T+10, 9T+6,$

$40T+10, 29T+6, 18T+9, 11T+5, 7T+8, 6T+4)$,
...,
$(46T+1, 2, 18T+4, 43T+3, 21T-1, 26T-2,$
$5T-1, 18T-2, 23T, 26T-1, 35T+1, 16T-1,$
$29T+1, 38T, 22T+1, 40T, 10T+1, 42T, 30T+1,$
$20T-1, 12T, 9T-2, 7T-1)$,
$(46T+1, 1, 18T+2, 43T+2, 21T, 26T, 5T, 18T,$
$23T+1, 26T+1, 35T+2, 16T+1, 29T+2, 38T+2,$
$22T+2, 40T+2, 10T+2, 42T+2, 32T+1, 24T+1,$
$12T+1, 9T, 7T)$ }.
Decompose this C_{23}-T-foil into s C_{23}-t-foils. Then these starters comprise a balanced C_{23}-t-foil decomposition of K_n.

Example 9.1. Balanced C_{23}-decomposition of K_{47}.
{(47, 1, 20, 45, 21, 26, 5, 18, 28, 34, 37, 17, 31, 40, 24, 42, 12, 44, 33, 25, 13, 9, 7)}.
This stater comprises a balanced C_{23}-decomposition of K_{47}.

Example 9.2. Balanced C_{23}-2-foil decomposition of K_{93}.
{(93, 2, 40, 89, 41, 50, 9, 34, 54, 66, 71, 31, 59, 76, 45, 80, 21, 84, 61, 39, 24, 16, 13),
(93, 1, 38, 88, 42, 52, 10, 36, 47, 53, 72, 33, 60, 78, 46, 82, 22, 86, 65, 49, 25, 18, 14)}.
This stater comprises a balanced C_{23}-2-foil decomposition of K_{93}.

Example 9.3. Balanced C_{23}-3-foil decomposition of K_{139}.
{(139, 3, 60, 133, 61, 74, 13, 50, 80, 98, 105, 45, 87, 112, 66, 118, 30, 124, 90, 57, 35, 23, 19),
(139, 2, 58, 132, 62, 76, 14, 52, 69, 77, 106, 47, 88, 114, 67, 120, 31, 126, 91, 59, 36, 25, 20),
(139, 1, 56, 131, 63, 78, 15, 54, 70, 79, 107, 49, 89, 116, 68, 122, 32, 128, 97, 73, 37, 27, 21)}.
This stater comprises a balanced C_{23}-3-foil decomposition of K_{139}.

Example 9.4. Balanced C_{23}-4-foil decomposition of K_{185}.
{(185, 4, 80, 177, 81, 98, 17, 66, 106, 130, 139, 59, 115, 148, 87, 156, 39, 164, 119, 75, 46, 30, 25),
(185, 3, 78, 176, 82, 100, 18, 68, 91, 101, 140, 61, 116, 150, 88, 158, 40, 166, 120, 77, 47, 32, 26),
(185, 2, 76, 175, 83, 102, 19, 70, 92, 103, 141, 63, 117, 152, 89, 160, 41, 168, 121, 79, 48, 34, 27),
(185, 1, 74, 174, 84, 104, 20, 72, 93, 105, 142, 65, 118, 154, 90, 162, 42, 170, 129, 97, 49, 36, 28)}.
This stater comprises a balanced C_{23}-4-foil decomposition of K_{185}.

Example 9.5. Balanced C_{23}-5-foil decomposition of K_{231}.
{(231, 5, 100, 221, 101, 122, 21, 82, 132, 162, 173, 73, 143, 184, 108, 194, 48, 204, 148, 93, 57, 37, 31),
(231, 4, 98, 220, 102, 124, 22, 84, 113, 125, 174, 75, 144, 186, 109, 196, 49, 206, 149, 95, 58, 39, 32),
(231, 3, 96, 219, 103, 126, 23, 86, 114, 127, 175, 77, 145, 188, 110, 198, 50, 208, 150, 97, 59, 41, 33),
(231, 2, 94, 218, 104, 128, 24, 88, 115, 129, 176, 79, 146, 190, 111, 200, 51, 210, 151, 99, 60, 43, 34),
(231, 1, 92, 217, 105, 130, 25, 90, 116, 131, 177, 81, 147, 192, 112, 202, 52, 212, 161, 121, 61, 45, 35)}.
This stater comprises a balanced C_{23}-5-foil decomposition of K_{231}.

Example 9.6. Balanced C_{23}-6-foil decomposition of K_{277}.
{(277, 6, 120, 265, 121, 146, 25, 98, 158, 194, 207, 87, 171, 220, 129, 232, 57, 244, 177, 111, 68, 44, 37),
(277, 5, 118, 264, 122, 148, 26, 100, 135, 149, 208, 89, 172, 222, 130, 234, 58, 246, 178, 113, 69, 46, 38),
(277, 4, 116, 263, 123, 150, 27, 102, 136, 151, 209, 91, 173, 224, 131, 236, 59, 248, 179, 115, 70, 48, 39),
(277, 3, 114, 262, 124, 152, 28, 104, 137, 153, 210, 93, 174, 226, 132, 238, 60, 250, 180, 117, 71, 50, 40),
(277, 2, 112, 261, 125, 154, 29, 106, 138, 155, 211, 95, 175, 228, 133, 240, 61, 252, 181, 119, 72, 52, 41),
(277, 1, 110, 260, 126, 156, 30, 108, 139, 157, 212, 97, 176, 230, 134, 242, 62, 254, 193, 145, 73, 54, 42)}.
This stater comprises a balanced C_{23}-6-foil decomposition of K_{277}.

Corollary 9.1. K_n has a balanced C_{23}-bowtie decomposition if and only if $n \equiv 1 \pmod{92}$.

Corollary 9.2. K_n has a balanced C_{23}-trefoil decomposition if and only if $n \equiv 1 \pmod{138}$.

10. Balanced C_{25}-foil designs

Theorem 10. K_n has a balanced C_{25}-t-foil decomposition if and only if $n \equiv 1 \pmod{50t}$.

Proof. **(Necessity)** Obvious by Theorem 1.
(Sufficiency) Put $n = 50st + 1, T = st$. Then $n = 50T + 1$.
Case 1. n = 51. (Example 10.1. Balanced C_{25}-decomposition of K_{51}.)
Case 2. n = 50T + 1, T ≥ 2. Construct a

C_{25}-T-foil as follows:
{ $(50T+1, T, 24T, 47T+1, 21T+1, 24T+2,$
$3T+1, 4T+2, 12T+2, 22T+3, 29T+3, 44T+4,$
$6T+3, 26T+4, 41T+4, 9T+4, T+3, 33T+4,$
$17T+3, 7T+3, 38T+3, 24T+3, 19T+2, 14T+2,$
$2T+1)$,
$(50T+1, T-1, 24T-2, 47T, 21T+2, 24T+4,$
$3T+2, 4T+4, 12T+3, 22T+5, 29T+4, 44T+6,$
$6T+4, 26T+6, 41T+5, 9T+6, T+4, 33T+6,$
$17T+4, 7T+5, 38T+4, 24T+5, 19T+3, 14T+4,$
$2T+2)$,
$(50T+1, T-2, 24T-2, 47T-1, 21T+3, 24T+6,$
$3T+3, 4T+6, 12T+4, 22T+7, 29T+5, 44T+8,$
$6T+5, 26T+8, 41T+6, 9T+8, T+5, 33T+8,$
$17T+5, 7T+7, 38T+5, 24T+7, 19T+4, 14T+6,$
$2T+3)$,
...,
$(50T+1, 3, 22T+6, 46T+4, 22T-2, 26T-4,$
$4T-2, 6T-4, 13T-1, 24T-3, 30T, 46T-2, 7T,$
$28T-2, 42T+1, 11T-2, 2T, 35T-2, 18T, 9T-3,$
$39T, 26T-3, 20T-1, 16T-4, 3T-2)$,
$(50T+1, 2, 22T+4, 46T+3, 22T-1, 26T-2,$
$4T-1, 6T-2, 13T, 24T-1, 30T+1, 46T, 7T+1,$
$28T, 42T+2, 11T, 44T, 35T, 18T+1, 9T-1,$
$39T+1, 26T-1, 20T, 16T-2, 3T-1)$,
$(50T+1, 1, 22T+2, 46T+2, 22T, 26T, 4T, 6T,$
$13T+1, 24T+1, 30T+2, 41T+3, 7T+2, 28T+2,$
$47T+2, 11T+2, 44T+1, 35T+2, 18T+2, 9T+1,$
$39T+2, 26T+1, 20T+1, 16T, 3T)$ }.
Decompose this C_{25}-T-foil into s C_{25}-t-foils. Then these starters comprise a balanced C_{25}-t-foil decomposition of K_n.

Example 10.1. Balanced C_{25}-decomposition of K_{51}.
{(51, 1, 24, 48, 22, 26, 4, 6, 14, 25, 32, 44, 9, 30, 49, 13, 46, 37, 20, 10, 41, 27, 21, 16, 3)}.
This stater comprises a balanced C_{25}-decomposition of K_{51}.

Example 10.2. Balanced C_{25}-2-foil decomposition of K_{101}.
{(101, 2, 48, 95, 43, 50, 7, 10, 26, 47, 61, 92, 15, 56, 86, 22, 88, 70, 37, 17, 79, 51, 40, 30, 5),
(101, 1, 46, 94, 44, 52, 8, 12, 27, 49, 62, 85, 16, 58, 96, 24, 89, 72, 38, 19, 80, 53, 41, 32, 6)}.
This stater comprises a balanced C_{25}-2-foil decomposition of K_{101}.

Example 10.3. Balanced C_{25}-3-foil decomposition of K_{151}.
{(151, 3, 72, 142, 64, 74, 10, 14, 38, 69, 90, 136, 21, 82, 127, 31, 6, 103, 54, 24, 117, 75, 59, 44, 7),
(151, 2, 70, 141, 65, 76, 11, 16, 39, 71, 91, 138, 22, 84, 128, 33, 132, 105, 55, 26, 118, 77, 60, 46, 8),
(151, 1, 68, 140, 66, 78, 12, 18, 40, 73, 92, 126, 23, 86, 143, 35, 133, 107, 56, 28, 119, 79, 61, 48, 9)}.
This stater comprises a balanced C_{25}-3-foil decomposition of K_{151}.

Example 10.4. Balanced C_{25}-4-foil decomposition of K_{201}.
{(201, 4, 96, 189, 85, 98, 13, 18, 50, 91, 119, 180, 27, 108, 168, 40, 7, 136, 71, 31, 155, 99, 78, 58, 9),
(201, 3, 94, 188, 86, 100, 14, 20, 51, 93, 120, 182, 28, 110, 169, 42, 8, 138, 72, 33, 156, 101, 79, 60, 10),
(201, 2, 92, 187, 87, 102, 15, 22, 52, 95, 121, 184, 29, 112, 170, 44, 176, 140, 73, 35, 157, 103, 80, 62, 11),
(201, 1, 90, 186, 88, 104, 16, 24, 53, 97, 122, 167, 30, 114, 190, 46, 177, 142, 74, 37, 158, 105, 81, 64, 12)}.
This stater comprises a balanced C_{25}-4-foil decomposition of K_{201}.

Example 10.5. Balanced C_{25}-5-foil decomposition of K_{251}.
{(251, 5, 120, 236, 106, 122, 16, 22, 62, 113, 148, 224, 33, 134, 209, 49, 8, 169, 88, 38, 193, 123, 97, 72, 11),
(251, 4, 118, 235, 107, 124, 17, 24, 63, 115, 149, 226, 34, 136, 210, 51, 9, 171, 89, 40, 194, 125, 98, 74, 12),
(251, 3, 116, 234, 108, 126, 18, 26, 64, 117, 150, 228, 35, 138, 211, 53, 10, 173, 90, 42, 195, 127, 99, 76, 13),
(251, 2, 114, 233, 109, 128, 19, 28, 65, 119, 151, 230, 36, 140, 212, 55, 220, 175, 91, 44, 196, 129, 100, 78, 14),
(251, 1, 112, 232, 110, 130, 20, 30, 66, 121, 152, 208, 37, 142, 237, 57, 221, 177, 92, 46, 197, 131, 101, 80, 15)}.
This stater comprises a balanced C_{25}-5-foil decomposition of K_{251}.

Example 10.6. Balanced C_{25}-6-foil decomposition of K_{301}.
{(301, 6, 144, 283, 127, 146, 19, 26, 74, 135, 177, 268, 39, 160, 250, 58, 9, 202, 105, 45, 231, 147, 116, 86, 13),
(301, 5, 142, 282, 128, 148, 20, 28, 75, 137, 178, 270, 40, 162, 251, 60, 10, 204, 106, 47, 232, 149, 117, 88, 14),
(301, 4, 140, 281, 129, 150, 21, 30, 76, 139, 179, 272, 41, 164, 252, 62, 11, 206, 107, 49, 233, 151, 118, 90, 15),
(301, 3, 138, 280, 130, 152, 22, 32, 77, 141, 180, 274, 42, 166, 253, 64, 12, 208, 108, 51, 234, 153, 119, 92, 16),
(301, 2, 136, 279, 131, 154, 23, 34, 78, 143, 181, 276, 43, 168, 254, 66, 264, 210, 109, 53, 235, 155, 120, 94, 17),
(301, 1, 134, 278, 132, 156, 24, 36, 79, 145, 182, 249, 44, 170, 284, 68, 265, 212, 110, 55, 236, 157, 121, 96, 18)}.
This stater comprises a balanced C_{25}-6-foil decomposition of K_{301}.

Corollary 10.1. K_n has a balanced C_{25}-bowtie decomposition if and only if $n \equiv 1 \pmod{100}$.

Corollary 10.2. K_n has a balanced C_{25}-trefoil decomposition if and only if $n \equiv 1 \pmod{150}$.

11. Conjectures

Conjecture 11. K_n has a balanced C_k-t-foil decomposition if and only if $n \equiv 1 \pmod{2kt}$.

Conjecture 11.1. K_n has a balanced C_k-bowtie decomposition if and only if $n \equiv 1 \pmod{4k}$.

Conjecture 11.2. K_n has a balanced C_k-trefoil decomposition if and only if $n \equiv 1 \pmod{6k}$.

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