ハミルトン C_k-Bowtie デザイン

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Hamilton C_k-Bowtie Designs

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In graph theory, the decomposition problem of graphs is a very important topic. Various type of decompositions of many graphs can be seen in the literature of gaph theory. This paper gives a Hamilton C_k-bowtie decomposition of the complete multi-grapf λK_n.

Key words: Hamilton C_k-bowtie decomposition, Complete multi-graph, Graph theory

1 Introduction

Let K_n denote *the complete graph* of n vertices. *The complete multi-graph* λK_n is the complete graph K_n in which every edge is taken λ times. Let C_k be *the k-cycle* (or *the cycle* on k vertices). *The C_k-bowtie* is a graph of 2 edge-disjoint C_k's with a common vertex and the common vertex is called *the center of the C_k-bowtie.* In particular, a C_k-bowtie satisfying $n = 2(k-1)+1$ is called *the Hamilton C_k-bowtie* because the C_k-bowtie spans λK_n.
When λK_n is decomposed into edge-disjoint sum of Hamilton C_k-bowties, we say that *λK_n has a Hamilton C_k-bowtie decomposition.* This Hamilton C_k-bowtie decomposition of λK_n is called *a Hamilton C_k-bowtie design.*

In this paper, it is shown that the necessary condition for the existence of a Hamilton C_k-bowtie decomposition of λK_n is (i) $n = 2(k-1)+1$ and (ii) $\lambda \equiv 0 \pmod{2k}$ for even k, $\lambda \equiv 0 \pmod{k}$ for odd k. Decomposition algorithms are also given.

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or $3 \pmod 6$. This decomposition is known as *a Steiner triple system.* See Colbourn and Rosa[2] and Wallis[15]. Horák and Rosa[3] proved that K_n has a C_3-bowtie decomposition if and only if $n \equiv 1$ or $9 \pmod{12}$. This decomposition is known as *a C_3-bowtie system.*
For the design theory, see Colbourn[1], Lindner[4], and Ushio[5]. For the graph decomposition, see Ushio[6-7], Ushio and Fujimoto[8-14].

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2 Hamilton C_k-bowtie decomposition of λK_n

Notation. We consider the vertex set V of λK_n as $V = \{1, 2, ..., n\}$. We denote a Hamilton C_k-bowtie passing through $v_1 - v_2 - v_3 - ... - v_k - v_1$, $v_1 - v_{k+1} - v_{k+2} - ... - v_{2k-1} - v_1$ by $H = (v_1, v_2, v_3, ..., v_k) \cup (v_1, v_{k+1}, v_{k+2}, ..., v_{2k-1})$.

Theorem 1. If λK_n has a Hamilton C_k-bowtie decomposition, then (i) $n = 2(k-1)+1$ and (ii) $\lambda \equiv 0 \pmod{2k}$ for even k, $\lambda \equiv 0 \pmod{k}$ for odd k.

Theorem 2. If λK_n has a Hamilton C_k-bowtie decomposition, then $(s\lambda)K_n$ has a Hamilton C_k-bowtie decomposition for every s.

Theorem F. (Fermat) Let p be prime and a be integer. Then $a^p \equiv a \pmod{p}$.

Corollaly F1. Let p be prime and $(a, p) = 1$. Then $a^{p-1} \equiv 1 \pmod{p}$.

Corollaly F2. Let p be prime and $(a, p) = 1$. Then $sa^{p-1} \equiv s \pmod{p}$ for $1 \le s \le p-1$.

Definition. When $sa^{n-1} \equiv s \pmod{n}$, let $a_i = \underline{sa^{i-1} \bmod n}$ $(i = 1, 2, ..., n)$ for $1 \le s \le n-1$. Find the first i $(i = 2, 3, ..., n)$ such that $a_i = s$. Put the i be L. Then the sequence $a_1(= s), a_2(= sa), a_3(= sa^2), ..., a_L(= s)$ is called *an L-orbit starting s.*
When there exist $(n-1)$ L-orbits starting $1, 2, ..., n-1$, we say that *n admits L-orbits.*

Note. Let p be prime. It is a widely known result that p admits p-orbits and that a is called *a primitive root w.r.t. mod p.* In particular, the least a denoted g is called *the least primitive root w.r.t mod p.*

Example F.1. (p, g) table.
$(p, g) = (2,1), (3,2), (5,2), (7,3), (11,2), (13,2), (17,3), (19,2), (23,5), (29,2), (31,3), (37,2),$

(41,6), (43,3), (47,5), (53,2), (59,2), (61,2), (67,2), (71,7), (73,5), (79,3), (83,2), (89,3), (97,5), (101,2), (103,5), (107,2), (109,6), (113,3), (127,3), (131,2), (137,3), (139,2), (149,2), (151,6), (157,5), (163,2), (167,5), (173,2), (179,2), (181,2), (191,19), (193,5), (197,2), (199,3), (211,2), (223,3), (227,2), (229,6), (233,3), (239,7), (241,7), (251,6), (257,3), (263,5), (269,2), (271,6), (277,5), (281,3), (283,3), (293,2), (307,5), (311,17), (313,10), (317,2), (331,3), (337,10), (347,2), (349,2), (353,3), (359,7), (367,6), (373,2), (379,2), (383,5), (389,2), (397,5), (401,3), (409,21), (419,2), (421,2), (431,7), (433,5), (439,15), (443,2), (449,3), (457,13), (461,2), (463,3), (467,2), (479,13), (487,3), (491,2), (499,7), (503,5), (509,2), (521,3), (523,2), (541,2), (547,2), (557,2), (563,2), (569,3), (571,3), (577,5), (587,2), (593,3), (599,7), (601,7), (607,3), (613,2), (617,3), (619,2), (631,3), (641,3), (643,11), (647,5), (653,2), (659,2), (661,2), (673,5), (677,2), (683,5), (691,3), (701,2), (709,2), (719,11), (727,5), (733,6), (739,3), (743,5), (751,3), (757,2), (761,6), (769,11), (773,2), (787,2), (797,2), (809,3), (811,3), (821,2), (823,3), (827,2), (829,2), (839,11), (853,2), (857,3), (859,2), (863,5), (877,2), (881,3), (883,2), (887,5), (907,2), (911,17), (919,7), (929,3), (937,5), (941,2), (947,2), (953,3), (967,5), (971,6), (977,3), (983,5), (991,6), (997,7).

Theorem 3. *Let n be prime. When $n = 2(k-1)+1$, $\lambda \equiv 0 \pmod{2k}$, and k even, λK_n has a Hamilton C_k-bowtie decomposition.*

Example 3.1. Hamilton C_4-bowtie of $8K_7$.
$(n,g) = (7,3)$
n-orbit : $1,3,2,6,4,5,1$.
$H = (7,1,3,2) \cup (7,6,4,5)$
$H = (7,3,2,6) \cup (7,4,5,1)$
$H = (7,2,6,4) \cup (7,5,1,3)$.
These 3 starters comprise a Hamilton C_4-bowtie decomposition of $8K_7$.

Example 3.2. Hamilton C_6-bowtie of $12K_{11}$.
$(n,g) = (11,2)$
n-orbit : $1,2,4,8,5,10,9,7,3,6,1$.
$H = (11,1,2,4,8,5) \cup (11,10,9,7,3,6)$
$H = (11,2,4,8,5,10) \cup (11,9,7,3,6,1)$
$H = (11,4,8,5,10,9) \cup (11,7,3,6,1,2)$
$H = (11,8,5,10,9,7) \cup (11,3,6,1,2,4)$
$H = (11,5,10,9,7,3) \cup (11,6,1,2,4,8)$.
These 5 starters comprise a Hamilton C_6-bowtie decomposition of $12K_{11}$.

Example 3.3. Hamilton C_{10}-bowtie of $20K_{19}$.
$(n,g) = (19,2)$
n-orbit : $1,2,4,8,16,13,7,14,9,18,17,15,11,3,6,12,5,10,1$.
$H = (19,1,2,4,8,16,13,7,14,9)$
$\cup (19,18,17,15,11,3,6,12,5,10)$
$H = (19,2,4,8,16,13,7,14,9,18)$
$\cup (19,17,15,11,3,6,12,5,10,1)$
$H = (19,4,8,16,13,7,14,9,18,17)$
$\cup (19,15,11,3,6,12,5,10,1,2)$
$H = (19,8,16,13,7,14,9,18,17,15)$
$\cup (19,11,3,6,12,5,10,1,2,4)$
$H = (19,16,13,7,14,9,18,17,15,11)$
$\cup (19,3,6,12,5,10,1,2,4,8)$
$H = (19,13,7,14,9,18,17,15,11,3)$
$\cup (19,6,12,5,10,1,2,4,8,16)$
$H = (19,7,14,9,18,17,15,11,3,6)$
$\cup (19,12,5,10,1,2,4,8,16,13)$
$H = (19,14,9,18,17,15,11,3,6,12)$
$\cup (19,5,10,1,2,4,8,16,13,7)$
$H = (19,9,18,17,15,11,3,6,12,5)$
$\cup (19,10,1,2,4,8,16,13,7,14)$.
These 9 starters comprise a Hamilton C_{10}-bowtie decomposition of $20K_{19}$.

Example 3.4. Hamilton C_{12}-bowtie of $24K_{23}$.
$(n,g) = (23,5)$
n-orbit : $1,5,2,10,4,20,8,17,16,11,9,22,18,21,13,19,3,15,6,7,12,14,1$.
11 starters comprise a Hamilton C_{12}-bowtie decomposition of $24K_{23}$.

Example 3.5. Hamilton C_{16}-bowtie of $32K_{31}$.
$(n,g) = (31,3)$
n-orbit: $1,3,9,27,19,26,16,17,20,29,25,13,8,24,10,30,28,22,4,12,5,15,14,11,2,6,18,23,7,21,1$.
15 starters comprise a Hamilton C_{16}-bowtie decomposition of $32K_{31}$.

Example 3.6. Hamilton C_{22}-bowtie of $44K_{43}$.
$(n,g) = (43,3)$
n-orbit: $1,3,9,27,38,28,41,37,25,32,10,30,4,12,36,22,23,26,35,19,14,42,40,34,16,5,15,2,6,18,11,33,13,39,31,7,21,20,17,8,24,29,1$.
21 starters comprise a Hamilton C_{22}-bowtie decomposition of $44K_{43}$.

Example 3.7. Hamilton C_{24}-bowtie of $48K_{47}$.
$(n,g) = (47,5)$
n-orbit : $1,5,25,31,14,23,21,11,8,40,12,13,18,43,27,41,17,38,2,10,3,15,28,46,42,22,16,33,24,26,36,39,7,35,34,29,4,20,6,30,9,45,37,44,32,19,1$.
23 starters comprise a Hamilton C_{24}-bowtie decomposition of $48K_{47}$.

Example 3.8. Hamilton C_{30}-bowtie of $60K_{59}$.
$(n,g) = (59,2)$
n-orbit : $1,2,4,8,16,32,5,10,20,40,21,42,25,50,41,23,46,33,7,14,28,56,53,47,35,11,22,44,29,58,57,55,51,43,27,54,49,39,19,38,17,34,9,18,36,13,26,52,45,31,3,6,12,24,48,37,15,30,1$.
29 starters comprise a Hamilton C_{30}-bowtie decomposition of $60K_{59}$.

Example 3.9. Hamilton C_{34}-bowtie of $68K_{67}$.
$(n,g)=(67,2)$
n-orbit : $1,2,4,8,16,32,64,61,55,43,19,38,9,18,36,5,10,20,40,13,26,52,37,7,14,28,56,45,23,46,25,50,33,66,65,63,59,51,35,3,6,12,24,48,29,58,49,31,62,57,47,27,54,41,15,30,60,53,39,11,22,44,21,42,17,34,1$.
33 starters comprise a Hamilton C_{34}-bowtie decomposition of $68K_{67}$.

Example 3.10. Hamilton C_{36}-bowtie of $72K_{71}$.
$(n,g)=(71,7)$
n-orbit : $1,7,49,59,58,51,2,14,27,47,45,31,4,28,54,23,19,62,8,56,37,46,38,53,16,41,3,21,5,35,32,11,6,42,10,70,64,22,12,13,20,69,57,44,24,26,40,67,43,17,48,52,9,63,15,34,25,33,18,55,30,68,50,66,36,39,60,65,29,61,1$.
35 starters comprise a Hamilton C_{36}-bowtie decomposition of $72K_{71}$.

Example 3.11. Hamilton C_{40}-bowtie of $80K_{79}$.
$(n,g)=(79,3)$
n-orbit : $1,3,9,27,2,6,18,54,4,12,36,29,8,24,72,58,16,48,65,37,32,17,51,74,64,34,23,69,49,68,46,59,19,57,13,39,38,35,26,78,76,70,52,77,73,61,25,75,67,43,50,71,55,7,21,63,31,14,42,47,62,28,5,15,45,56,10,30,11,33,20,60,22,66,40,41,44,53,1$.
39 starters comprise a Hamilton C_{40}-bowtie decomposition of $80K_{79}$.

Example 3.12. Hamilton C_{42}-bowtie of $84K_{83}$.
$(n,g)=(83,2)$
n-orbit : $1,2,4,8,16,32,64,45,7,14,28,56,29,58,33,66,49,15,30,60,37,74,65,47,11,22,44,5,10,20,40,80,77,71,59,35,70,57,31,62,41,82,81,79,75,67,51,19,38,76,69,55,27,54,25,50,17,34,68,53,23,46,9,18,36,72,61,39,78,73,63,43,3,6,12,24,48,13,26,52,21,42,1$.
41 starters comprise a Hamilton C_{42}-bowtie decomposition of $84K_{83}$.

Theorem 4. Let n be prime. When $n=2(k-1)+1$, $\lambda\equiv 0 \pmod k$, and k odd, λK_n has a Hamilton C_k-bowtie decomposition.

Example 4.1. Hamilton C_3-bowtie of $3K_5$.
$(n,g)=(5,2)$
n-orbit : $1,2,4,3,1$.
$L_1 : 1,4,1 \quad L_2 : 2,3,2$.
$H=(5,1,4)\cup(5,2,3)$.
This starter comprises a Hamilton C_3-bowtie decomposition of $3K_5$.

Example 4.2. Hamilton C_7-bowtie of $7K_{13}$.
$(n,g)=(13,2)$
n-orbit : $1,2,4,8,3,6,12,11,9,5,10,7,1$.
$L_1 : 1,4,3,12,9,10,1 \quad L_2 : 2,8,6,11,5,7,2$.
$H=(13,1,4,3,12,9,10)\cup(13,2,8,6,11,5,7)$
$H=(13,4,3,12,9,10,1)\cup(13,8,6,11,5,7,2)$
$H=(13,3,12,9,10,1,4)\cup(13,6,11,5,7,2,8)$.
These 3 starters comprise a Hamilton C_7-bowtie decomposition of $7K_{13}$.

Example 4.3. Hamilton C_9-bowtie of $9K_{17}$.
$(n,g)=(17,3)$
n-orbit : $1,3,9,10,13,5,15,11,16,14,8,7,4,12,2,6,1$.
$L_1 : 1,9,13,15,16,8,4,2,1$
$L_2 : 3,10,5,11,14,7,12,6,3$.
$H=(17,1,9,13,15,16,8,4,2)$
$\cup(17,3,10,5,11,14,7,12,6)$
$H=(17,9,13,15,16,8,4,2,1)$
$\cup(17,10,5,11,14,7,12,6,3)$
$H=(17,13,15,16,8,4,2,1,9)$
$\cup(17,5,11,14,7,12,6,3,10)$
$H=(17,15,16,8,4,2,1,9,13)$
$\cup(17,11,14,7,12,6,3,10,5)$.
These 4 starters comprise a Hamilton C_9-bowtie decomposition of $9K_{17}$.

Example 4.4. Hamilton C_{15}-bowtie of $15K_{29}$.
$(n,g)=(29,2)$
n-orbit : $1,2,4,8,16,3,6,12,24,19,9,18,7,14,28,27,25,21,13,26,23,17,5,10,20,11,22,15,1$.
7 starters comprise a Hamilton C_{15}-bowtie decomposition of $15K_{29}$.

Example 4.5. Hamilton C_{19}-bowtie of $19K_{37}$.
$(n,g)=(37,2)$
n-orbit : $1,2,4,8,16,32,27,17,34,31,25,13,26,15,30,23,9,18,36,35,33,29,21,5,10,20,3,6,12,24,11,22,7,14,28,19,1$.
9 starters comprise a Hamilton C_{19}-bowtie decomposition of $19K_{37}$.

Example 4.6. Hamilton C_{21}-bowtie of $21K_{41}$.
$(n,g)=(41,6)$
n-orbit : $1,6,36,11,25,27,39,29,10,19,32,28,4,24,21,3,18,26,33,34,40,35,5,30,16,14,2,12,31,22,9,13,37,17,20,38,23,15,8,7,1$.
10 starters comprise a Hamilton C_{21}-bowtie decomposition of $21K_{41}$.

Example 4.7. Hamilton C_{27}-bowtie of $27K_{53}$.
$(n,g)=(53,2)$
n-orbit : $1,2,4,8,16,32,11,22,44,35,17,34,15,30,7,14,28,3,6,12,24,48,43,33,13,26,52,51,49,45,37,21,42,31,9,18,36,19,38,23,46,39,25,50,47,41,29,5,10,20,40,27,1$.
13 starters comprise a Hamilton C_{27}-bowtie decomposition of $27K_{53}$.

Example 4.8. Hamilton C_{31}-bowtie of $31K_{61}$.
$(n,g)=(61,2)$
n-orbit : $1,2,4,8,16,32,3,6,12,24,48,35,9,18,36,11,22,44,27,54,47,33,5,10,20,40,19,38,15,30,60,59,57,53,45,29,58,55,49,37,13,26,52,43,25,50,39,17,$

34, 7, 14, 28, 56, 51, 41, 21, 42, 23, 46, 31, 1.
15 starters comprise a Hamilton C_{31}-bowtie decomposition of $31K_{61}$.

Example 4.9. Hamilton C_{37}-bowtie of $37K_{73}$.
$(n, g) = (73, 5)$
n-orbit : 1, 5, 25, 52, 41, 59, 3, 15, 2, 10, 50, 31, 9, 45, 6, 30, 4, 20, 27, 62, 18, 17, 12, 60, 8, 40, 54, 51, 36, 34, 24, 47, 16, 7, 35, 29, 72, 68, 48, 21, 32, 14, 70, 58, 71, 63, 23, 42, 64, 28, 67, 43, 69, 53, 46, 11, 55, 56, 61, 13, 65, 33, 19, 22, 37, 39, 49, 26, 57, 66, 38, 44, 1.
18 starters comprise a Hamilton C_{37}-bowtie decomposition of $37K_{73}$.

Example 4.10. Hamilton C_{45}-bowtie of $45K_{89}$.
$(n, g) = (89, 3)$
n-orbit : 1, 3, 9, 27, 81, 65, 17, 51, 64, 14, 42, 37, 22, 66, 20, 60, 2, 6, 18, 54, 73, 41, 34, 13, 39, 28, 84, 74, 44, 43, 40, 31, 4, 12, 36, 19, 57, 82, 68, 26, 78, 56, 79, 59, 88, 86, 80, 62, 8, 24, 72, 38, 25, 75, 47, 52, 67, 23, 69, 29, 87, 83, 71, 35, 16, 48, 55, 76, 50, 61, 5, 15, 45, 46, 49, 58, 85, 77, 53, 70, 32, 7, 21, 63, 11, 33, 10, 30, 1.
22 starters comprise a Hamilton C_{45}-bowtie decomposition of $45K_{89}$.

Example 4.11. Hamilton C_{49}-bowtie of $49K_{97}$.
$(n, g) = (97, 5)$
n-orbit: 1, 5, 25, 28, 43, 21, 8, 40, 6, 30, 53, 71, 64, 29, 48, 46, 36, 83, 27, 38, 93, 77, 94, 82, 22, 13, 65, 34, 73, 74, 79, 7, 35, 78, 2, 10, 50, 56, 86, 42, 16, 80, 12, 60, 9, 45, 31, 58, 96, 92, 72, 69, 54, 76, 89, 57, 91, 67, 44, 26, 33, 68, 49, 51, 61, 14, 70, 59, 4, 20, 3, 15, 75, 84, 32, 63, 24, 23, 18, 90, 62, 19, 95, 87, 47, 41, 11, 55, 81, 17, 85, 37, 88, 52, 66, 39, 1.
24 starters comprise a Hamilton C_{49}-bowtie decomposition of $49K_{97}$.

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