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Separation of Two-Variable Reactance Sections

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A transformation which converts a multivariable positive-real function to another multivariable positive-real function is presented, and the conditions for reduction in degrees of the resulting positive-real function are discussed. Its application to the synthesis of a class of multivariable driving-point functions is discussed.

Key words: Circuit theory, Cascade synthesis, Lumped and distributed mixed networks, Multivariable positive-real functions, Richards' section.

1 Introduction

Multivariable positive-real function and matrix were introduced concerning the problem of designing a passive network composed of lumped elements with variable parameters [1]. The concept of such function and matrix was later applied to the problem of synthesizing passive networks composed of lossless transmission lines and lumped elements [2]. The above-mentioned synthesis problem has been discussed by several authors [3-12]. Koga made clear that an arbitrarily prescribed nxn positive-real matrix of several variables is realizable as the impedance or admittance matrix of a finite passive n-port by using the corresponding variable elements [5].

However, although the cascade synthesis problem of one-variable case, as is well known, is solved, multivariable case has not as yet been solved generally.

This paper presents two theorems with respect to the extraction of two-variable Richards' sections which are used for realizing a cascade synthesis and indicates the application to synthesis of a class of reciprocal networks by Bott-Duffin procedure. The results presented here are the extension of one of results given in [10].

We shall first give mathematical notations and symbols frequently used in this paper. The symbols $p_1, p_2, q_1, ..., q_n$ will be used as complex variables. The set of complex variables $q_1, ..., q_n$ is expressed as $\boldsymbol{q}$, i.e., $\boldsymbol{q} = (q_1, ..., q_n)$, and $\mathrm{Re}\, q_k \geqq 0$ $(k = 1, 2.., n)$ as $\mathrm{Re}\, \boldsymbol{q} \geqq 0$, etc..

2 Transformation for a Positive-real Function

In this section a transformation for positive-real functions is presented in Theorem 2. Under certain conditions the degree of the new positive-real function is lower than that of the original function, and this can be used to be obtain one step in the synthesis of the original function as illustrated in the next section.

Theorem 1: Let $f_1(p_1, p_2)$ be a two-variable polynomial given by

$$f_1(p_1,p_2) = \alpha\left\{g(p_2)+\varepsilon_1 g(-p_2)\right\}p_1^2 + \beta g(p_2)p_1 + \gamma\left\{g(p_2)+\varepsilon_1 g(-p_2)\right\} \quad (1)$$

or

$$f_1(p_1,p_2) = \alpha g(p_2)p_1^2 + \beta\left\{g(p_2)+\varepsilon_1 g(-p_2)\right\}p_1 + \gamma g(p_2) \quad (2)$$

where $g(p_2)$ is a monic Hurwitz polynomial, $\alpha, \beta, \gamma > 0$, and $\varepsilon_1 = \pm 1$, then the following three hold.

[1] The polynomial $f_1(p_1, p_2)$ is a Hurwitz polynomial.

[2] A rational function $w_1(p_1, p_2)$ defined by

$$w_1(p_1,p_2) = R_0\frac{f_1(p_1,p_2)+\varepsilon_0 f_1(-p_1,-p_2)}{f_1(p_1,p_2)-\varepsilon_0 f_1(-p_1,-p_2)} \quad (3)$$
$$(\varepsilon_0 = \pm 1)$$

is a reactance function for R_0 being a positive constant.

[3] A rational function $s_1(p_1, p_2)$ defined by

$$s_1(p_1,p_2) = \frac{f_1(-p_1,-p_2)}{f_1(p_1,p_2)}$$

is a bounded real function.

Proof: We shall begin with proving [2]. For $f_1(p_1, p_2)$ given by eqn.(1) we consider $w_1(p_1, p_2)$. From eqn.(1) we can get

$$f_1(-p_1,-p_2) = \varepsilon_1\alpha\left\{g(p_2)+\varepsilon_1 g(-p_2)\right\}p_1^2 - \beta g(-p_2)p_1 + \varepsilon_1\gamma\left\{g(p_2)+\varepsilon g(-p_2)\right\}. \quad (4)$$

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Substituting eqns.(1) and (4) into eqn.(3) yields the following.

$$\frac{w_1(p_1,p_2)}{R_0} = \begin{cases} \dfrac{2\alpha}{\beta}p_1+\dfrac{2\gamma}{\beta p_1}+\dfrac{g(p_2)-\varepsilon_1 g(-p_2)}{g(p_2)+\varepsilon_1 g(-p_2)} & (\varepsilon_0\varepsilon_1=1) \\ \dfrac{1}{\dfrac{2\alpha}{\beta}p_1+\dfrac{2\gamma}{\beta p_1}+\dfrac{g(p_2)-\varepsilon_1 g(-p_2)}{g(p_2)+\varepsilon_1 g(-p_2)}} & (\varepsilon_0\varepsilon_1=-1) \end{cases} \tag{5}$$

For $f_1(p_1,p_2)$ given by eqn.(2),

$$f_1(-p_1,-p_2)=\alpha g(-p_2)p_1^2 - \varepsilon\beta\left\{g(p_2)+\varepsilon g(-p_2)\right\}p_1+\gamma g(-p_2). \tag{6}$$

Substituting eqns.(2) and (4) into eqn.(3) yields the following.

$$\frac{w_1(p_1,p_2)}{R_0} = \begin{cases} \dfrac{1}{\dfrac{1}{\dfrac{\alpha}{2\beta}p_1+\dfrac{\gamma}{2\beta p_1}}+\dfrac{g(p_2)-\varepsilon_1 g(-p_2)}{g(p_2)+\varepsilon_1 g(-p_2)}} & (\varepsilon_0\varepsilon_1=1) \\ \dfrac{1}{\dfrac{\alpha}{2\beta}p_1+\dfrac{\gamma}{2\beta p_1}}+\dfrac{g(p_2)-\varepsilon_1 g(-p_2)}{g(p_2)+\varepsilon_1 g(-p_2)} & (\varepsilon_0\varepsilon_1=-1) \end{cases} \tag{7}$$

In eqns.(5) and (7), a rational function of p_1,

$$\frac{\alpha}{\beta}p_1+\frac{\gamma}{\beta p_1}$$

is a reactance function, and a rational function of p_2,

$$\frac{g(p_2)-\varepsilon_1 g(-p_2)}{g(p_2)+\varepsilon_1 g(-p_2)}$$

is a reactance function, because $g(p_2)$ is a Hurwitz polynomial of p_2. Therefore, $w_1(p_1,p_2)$ is the reactance function of p_1 and p_2. Thus, we complete the proof of [2].

To prove [1], we shall consider a rational function

$$\frac{w_1(p_1,p_2)}{R_0}+1.$$

The above is a two-variable positive-real function and can be written as the following right side.

$$\frac{w_1(p_1,p_2)}{R_0}+1=\frac{2f_1(p_1,p_2)}{f_1(p_1,p_2)-f_1(-p_1,-p_2)}$$

Therefore, $f_1(p_1,p_2)$ is the Hurwitz polynomial of p_1 and p_2.

Finally, we shall consider [3]. Due to [1], $s_1(p_1,p_2)$ is holomorphic in the open domain $\mathrm{Re}\,p_k>0$ $(k=1,2)$. Furthermore,

$$|s_1(p_1,p_2)|=1$$

for $\mathrm{Re}\,p_k=0$ $(k=1,2)$, except at singularities of indeterminacy. Thus, we see from the maximum modulus theorem that

$$|s_1(p_1,p_2)|<1$$

in the open domain $\mathrm{Re}\,p_k>0$ $(k=1,2)$. Q.E.D.

Theorem 2: Let $w(p_1,p_2,\boldsymbol{q})$ be a positive-real function, and assume that $w(p_1,p_2,\boldsymbol{q})+w(-p_1,-p_2,\boldsymbol{q})$ is factorizable as

$$w(p_1,p_2,\boldsymbol{q})+w(-p_1,-p_2,-\boldsymbol{q}) = f_1(p_1,p_2)f_1(-p_1,-p_2)\ \zeta(p_1,p_2,\boldsymbol{q}) \tag{8}$$

where $f_1(p_1,p_2)$ is the polynomials of eqns.(1) or (2).

If there exists a positive constant R_0 which satisfies

$$w(-\varphi_{1k}(-p_2),-p_2,\boldsymbol{q}) \underset{q}{\equiv} R_0 \quad (k=1,2) \tag{9}$$

for the branch functions $p_1=\varphi_{1k}(-p_2)(k=1,2)$ of an algebraic function defined by $f_1(-p_1,-p_2)=0$, then a rational function

$$w_2(p_1,p_2,\boldsymbol{q})=\frac{w_1(p_1,p_2)w(p_1,p_2,\boldsymbol{q})-R_0^2}{w_1(p_1,p_2)-w(p_1,p_2,\boldsymbol{q})} \tag{10}$$

is also positive real for $w_1(p_1,p_2)$ given in eqn.(3), and

$$\deg_{p_k} w_2=\deg_{p_k} w-\deg_{p_k} f_1 \quad (k=1,2), \tag{11}$$

$$\deg_{q_k} w_2=\deg_{q_k} w \quad (k=1,2\ldots,n). \tag{12}$$

Proof: Let $s(p_1,p_2,\boldsymbol{q}), s_1(p_1,p_2)$ and $s_2(p_1,p_2,\boldsymbol{q})$, respectively, be defined as

$$s(p_1,p_2,\boldsymbol{q})=\frac{w(p_1,p_2,\boldsymbol{q})-R_0}{w(p_1,p_2,\boldsymbol{q})+R_0},$$

$$s_1(p_1,p_2)=\frac{w_1(p_1,p_2)-R_0}{w_1(p_1,p_2)+R_0}$$

and

$$s_2(p_1,p_2,\boldsymbol{q})=\frac{w_2(p_1,p_2,\boldsymbol{q})-R_0}{w_2(p_1,p_2,\boldsymbol{q})+R_0}.$$

The function $s_2(p_1,p_2,\boldsymbol{q})$ can be written as

$$s_2(p_1,p_2,\boldsymbol{q})=\frac{s(p_1,p_2,\boldsymbol{q})}{s_1(p_1,p_2)} = \frac{f_1(p_1,p_2)}{f_1(-p_1,-p_2)}\cdot\frac{w(p_1,p_2,\boldsymbol{q})-R_0}{w(p_1,p_2,\boldsymbol{q})+R_0}. \tag{13}$$

Because the function $w(p_1,p_2,\boldsymbol{q})-R_0$ has the factor $(p_1-\varphi_{11}(-p_2))\ (p_1-\varphi_{12}(-p_2))$, $f_1(-p_1,-p_2)$ is canceled from the denominator and the numerator of

$s_2(p_1, p_2, \boldsymbol{q})$. Therefore, $s_2(p_1, p_2, \boldsymbol{q})$ is holomorphic in the domain $\mathrm{Re}\, p_1 > 0, \mathrm{Re}\, p_2 > 0$ and $\mathrm{Re}\, \boldsymbol{q} > 0$. Furthermore,

$$|s_2(p_1, p_2, \boldsymbol{q})| = 1$$

for $\mathrm{Re}\, p_k = 0\ (k = 1, 2)$ and $\mathrm{Re}\, q = 0$, except at singularities of indeterminacy. Therefore $s_2(p_1, p_2, \boldsymbol{q})$ is bounded real, and

$$\deg_{pk} s_2 = \deg_{pk} s \quad (k = 1, 2).$$
$$\deg_{qk} s_2 = \deg_{qk} s \quad (k = 1, 2, ..., n).$$

Thus, $w_2(p_1, p_2, \boldsymbol{q})$ is positive real. From eqns.(8) and (9),

$$w(\varphi_{1k}(p_2), p_2, \boldsymbol{q}) + R_0 \underset{q}{\equiv} 0 \quad (k{=}1, 2).$$

In eqn.(13) the factor $f_1(p_1, p_2)$ in the denominator of $s_2(p_1, p_2, q)$ is exactly disable from its numerator. Therefore,

$$\deg_{p_k} s_2 = \deg_{p_k} s - \deg_{p_k} f_1 \qquad (\ k = 1, 2)$$
$$\deg_{q_k} s_2 = \deg_{q_k} s \qquad (\ k = 1, 2, ..., n)\,.$$

Thus, from the above results we see eqns.(11) and (12) are true. Q.E.D.

3 Application to Network Synthesis

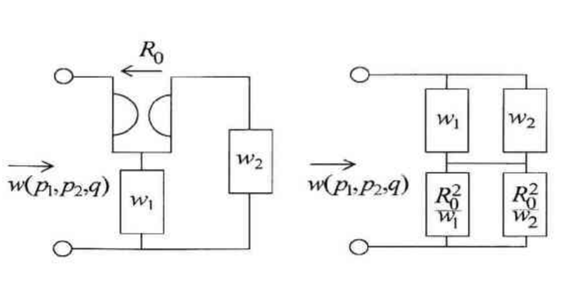
The transformation of Section 2 can be used to obtain one step in the synthesis of a class of networks by Bott-Duffin type procedure or alternatively by using gyrators.

In this section we shall realize the positive-real function $w(p_1, p_2, \boldsymbol{q})$ which satisfies the conditions of Theorem 2. In order to do this we assume $w(p_1, p_2, \boldsymbol{q})$ is the impedance function.

From eqn.(10) $w(p_1, p_2, \boldsymbol{q})$ can be written as follows:

$$w(p_1, p_2, \boldsymbol{q}) = \frac{w_1(p_1, p_2) w_2(p_1, p_2, \boldsymbol{q}) + R_0^2}{w_1(p_1, p_2) + w_2(p_1, p_2, \boldsymbol{q})}$$

The above $w(p_1, p_2, \boldsymbol{q})$ is the driving point impedance function of each of circuits shown in Fig.1.



(a) Richards Section. (b) Butt-Duffin Procedure.

Fig.1 Realizations of $w(p_1, p_2, \boldsymbol{q})$.

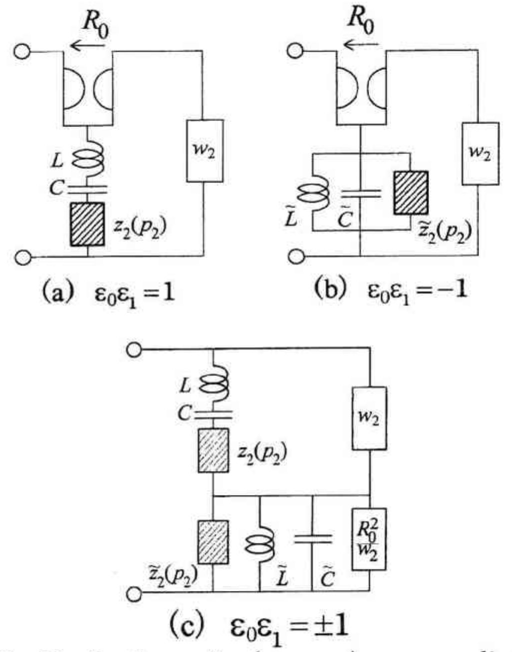


Fig.2 Realizations of $w(p_1, p_2, \boldsymbol{q})$ coresponding to eqn.(5).

This means that there are two types of networks with respect to $w(p_1, p_2, \boldsymbol{q})$ given. Furthermore we can have two types of networks with respect to realizing the reactance function $w_1(p_1, p_2)$ for $\varepsilon_0 = \pm 1$, as will be seen from Fig.2(a),(b) or Fig.3(a),(b).

Fig.1(a) shows the extraction of Richards' section having the gyrator in which the ratio is R_0. Fig.1(b) shows one step of the Bott-Duffin synthesis processes.

Therefore, if the impedance $w_1(p_1, p_2)$ can be specified, $w(p_1, p_2, \boldsymbol{q})$ is then reduced to the realization of two impedances ($w_2(p_1, p_2, \boldsymbol{q})$ and its dual $R_0^2/w_2(p_1, p_2, \boldsymbol{q})$).

In below the realization of $w_1(p_1, p_2)$ corresponding to each of the polynomials shown in eqns.(5) and (7) is given.

Realizations corresponding to eqns.(5) and (7) are shown in Figs.2 and 3, respectively. The symbols L and C, etc., in these figures, express element values of a p_1-variable inductor and a p_1-variable capacitor, respectively, and these are given below. Furthermore, $z_2(p_2)$ and $\tilde{z}_2(p_2)$ represent driving point impedances of circuits consisting of p_2-variable elements, as shown below.

For Fig.2:

$$L = \frac{2R_0\alpha}{\beta},\ \ \tilde{L} = \frac{\beta R_0}{2\gamma},\ C = \frac{\beta}{2R_0\gamma},\ \tilde{C} = \frac{2\alpha}{\beta R_0},$$

$$z_2(p_2) = R_0 \frac{g(p_2) - \varepsilon_1 g(-p_2)}{g(p_2) + \varepsilon_1 g(-p_2)}, \quad \tilde{z}_2(p_2) = \frac{R_0^2}{z_2(p_2)}$$

For Fig.3:

$$L = \frac{R_0\alpha}{2\beta},\ \ \tilde{L} = \frac{2\beta R_0}{\gamma},\ C = \frac{2\beta}{R_0\gamma},\ \tilde{C} = \frac{\alpha}{2\beta R_0}$$

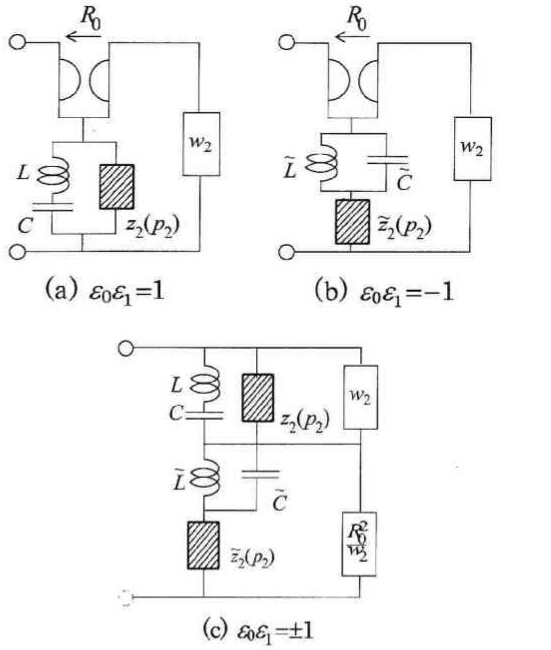


Fig.3 Realizations of $w(p_1, p_2, \boldsymbol{q})$ coresponding to eqn.(7).

$$z_2(p_2) = R_0 \frac{g(p_2) + \varepsilon_1 g(-p_2)}{g(p_2) - \varepsilon_1 g(-p_2)}, \quad \tilde{z}_2(p_2) = \frac{R_0^2}{z_2(p_2)}$$

4 Example

Realize a three-variable positive-real function $w(p_1, p_2, q_1)$ given by

Now, define $g(p_2)$ by

$$g(p_2) = p_2 + \frac{1}{4},$$

then this is a Hurwitz polynomial. Thus,

$$g(p_2) - g(-p_2) = 2p_2,$$
$$g(p_2) + g(-p_2) = \frac{1}{2}.$$

In addition,

$$\alpha = 4, \quad \beta = 16, \quad \gamma = 8, \text{ and } \varepsilon_1 = 1.$$

Thus we see that $f_1(p_1, p_2)$ satisfies the conditions given in Theorem 1(eqn.(2)). The gyrator of ratio R_0 is given as

$$R_0 \equiv 2$$

for roots

$$p_1 = -\frac{2 \pm \sqrt{2 - 16\,(2p_2 - 1)\,p_2}}{4p_2 - 1}$$

of $f_1(-p_1, -p_2) = 0$ for the open domain except at $\mathrm{Re}\, p_k > 0 (k = 1, 2)$. Therefore, $w_1(p_1, p_2)$ yielded from eqn.(3) becomes

$$w_1(p_1, p_2) = \begin{cases} \dfrac{p_1^2 + 2}{2\,(p_2 p_1^2 + p_1 + 2p_2)} & (\varepsilon_0 = 1) \\[2ex] \dfrac{8\,(p_2 p_1^2 + p_1 + 2p_2)}{p_1^2 + 2} & (\varepsilon_0 = -1)\,. \end{cases}$$

$$w(p_1, p_2, q_1) = \frac{n(p_1, p_2, q_1)}{m(p_1, p_2, q_1)}$$
$$= \frac{4\left\{2\left(p_1^2+2\right)p_2 + p_1^3 + p_1^2 + 4p_1 + 2\right\}q_1^2 + \left\{8\left(p_1^2+2\right)(p_1+1)\,p_2 + 9p_1^2 + 8p_1 + 2\right\}q_1 + 8\left(p_1^2+2\right)p_2 + p_1^3 + p_1^2 + 10p_1 + 2}{\left\{8\left(p_1^2+2\right)(p_1+1)\,p_2 + 9p_1^2 + 8p_1 + 2\right\}q_1^2 + \left\{2\left(p_1^2+2\right)p_2 + p_1^3 + p_1^2 + 4p_1 + 2\right\}q_1 + 2\left(p_1^2+2\right)(p_1+1)\,p_2 + 3p_1^2 + 2p_1 + 2}.$$

For the above $w(p_1, p_2, q_1)$, eqn.(8) becomes

$$w(p_1, p_2, q_1) + w(-p_1, -p_2, -q_1)$$
$$= -\frac{2\left(2q_1^2+1\right)^2 \left\{(4p_2+1)\,p_1^2 + 4p_1 + 2\,(4p_2+1)\right\} \left\{(1-4p_2)\,p_1^2 - 4p_1 + 2\,(1-4p_2)\right\}}{m(p_1, p_2, q_1)\,m(-p_1, -p_2, -q_1)}.$$

Let $f_1(p_1, p_2)$ be

$$f_1(p_1, p_2) = 4\left(p_2 + \frac{1}{4}\right)p_1^2 + 4p_1 + 8\left(p_2 + \frac{1}{4}\right).$$

Therefore

$$f_1(-p_1, -p_2) = 4\left(\frac{1}{4} - p_2\right)p_1^2 - 4p_1 + 8\left(\frac{1}{4} - p_2\right).$$

Furthermore, $w_2(p_1, p_2, q_1)$, which can be obtained from eqn.(10), is

$$w_2(p_1, p_2, q_1) = \begin{cases} \dfrac{4\,(p_1+1)\,q_1^2 + q_1 + p_1 + 1}{q_1^2 + (p_1+1)\,q_1 + 1} & (\varepsilon_0 = 1) \\[2ex] \dfrac{4\left\{q_1^2 + (p_1+1)\,q_1 + 1\right\}}{4\,(p_1+1)\,q_1^2 + q_1 + p_1 + 1} & (\varepsilon_0 = -1)\,. \end{cases}$$

Fig.4 shows three realizations of $w(p_1, p_2, q_1)$ given in this section.

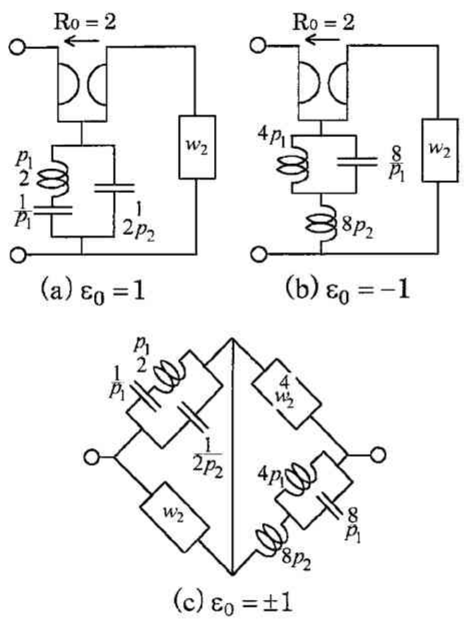


Fig.4 Realizations of $w(p_1, p_2, q_1)$.

5 Conclusions

A new transformation for a class of multivariable positive-real functions has been presented. This transformation has been applied to obtain a Bott-Duffin type cycle in the synthesis of functions which may otherwise be difficult to realize. An alternate realization is obtained using a gyrator.

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