超幾何微分方程式に付随するリッカチ方程式についての一注意

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A remark on the Riccati equation associated with a hypergeometric equation

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The Riccati equation associated with a hypergeometric equation is solved by using expansion with respect to parameters. This construction of solution provides a point of view about the Gauss formula which gives the special value of hypergeometric function at the unit.

Key words: Hypergeometric function, Riccati equation, Riemann zeta values

1. Introduction

We consider the following hypergeometric differential equation:

$$t(1-t)\frac{d^2w}{dt^2}+\{1-z-(x+y+1)t\}\frac{dw}{dt}-xyw=0, \quad (1)$$

where x, y and z are complex constants. The classical hypergeometric series

$$F(x,y,1-z;t)=\sum_{n=0}^{\infty}\frac{(x)_n(y)_n}{(1-z)_n n!}t^n \quad (2)$$

defines a holomorphic solution to (1) at the origin. Here $(x)_n = x(x+1)\cdots(x+n-1)$ denotes the Pochhammer symbol. In this article, we construct a solution to (1) in the form

$$w=\exp\left(\int_0^t S(x,y,z;u)du\right) \quad (3)$$

with S having an expansion

$$S(x,y,z;t)=\sum_{n=2}^{\infty}S_n(t), \quad (4)$$

where $S_n(t) = S_n(x,y,z;t)$ are homogeneous polynomials of x, y, z of degree n. We revisit the Gauss-Kummer formula (cf. [1]) for the value of hypergeometric function at the unit:

$$F(x,y,1-z;1)=\frac{\Gamma(1-z)\Gamma(1-x-y-z)}{\Gamma(1-x-z)\Gamma(1-y-z)}. \quad (5)$$

Combining this and the expression

$$\Gamma(1-x)=\exp\left(\gamma x+\sum_{n=0}^{\infty}\frac{\zeta(n)}{n}x^n\right), \quad (6)$$

we have another form of the Gauss formula:

$$F(x,y,1-z;1)=\exp\left(\sum_{n=2}^{\infty}\frac{\zeta(n)}{n}(z^n+(x+y+z)^n-(x+z)^n-(y+z)^n)\right). \quad (7)$$

Here $\zeta(n)$ denote the Riemann zeta values. The aim of this article is to interpret the Gauss-Kummer formula to the Riccati equation for hypergeometric equation and obtain an intriguing property of S which is obtained by solving the system of infinitely many differential equations coming from the Riccati equation. We obtain (7) directly without using (6). Our proof is based on an idea given in [2].

2. Riccati equation corresponding to hypergeometric differential equation and its solution

Putting (3) into (1), we have the following Riccati equation for S:

$$t(1-t)\left(\frac{dS}{dt}+S^2\right)+\{1-z-(x+y+1)t\}S-xy=0. \quad (8)$$

If S has an expansion of the form (4), each homogeneous part S_n of degree n with respect to x, y and z should satisfy the following system of differential equations:

$$t(1-t)\frac{dS_2}{dt}+(1-t)S_2-xy=0, \quad (9)$$

$$t(1-t)\frac{dS_3}{dt}+(1-t)S_3-((x+y)t+z)S_2=0, \quad (10)$$

$$t(1-t)\frac{dS_n}{dt}+(1-t)S_n-((x+y)t+z)S_{n-1}-t(1-t)\sum_{j=2}^{n-2}S_jS_{n-j}=0 \quad (n\geq 4) \quad (11)$$

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with initial data

$$S_n(0) = xyz^{n-2} \tag{12}$$

for $n \geq 2$. This system can be solved recursively and we have

$$S_2(t) = -\frac{xy\log(1-t)}{t}, \tag{13}$$

$$\begin{aligned} S_3(t) = (x+y+z)xy \sum_{m_1 \geq m_2 \geq 1} \frac{t^{m_1-1}}{m_1 m_2} \\ -(x+y)xy \sum_{m=1}^{\infty} \frac{t^{m-1}}{m^2}, \end{aligned} \tag{14}$$

$$\begin{aligned} S_n(t) = \frac{x+y+z}{t} \int_0^t \frac{S_{n-1}(u)}{1-u} du - \frac{x+y}{t} \times \\ \int_0^t S_{n-1}(u)du - \frac{1}{t} \int_0^t u \sum_{m=2}^{n-2} S_m(u) S_{n-m}(u) du \end{aligned} \tag{15}$$

for $n \geq 4$.

Theorem 1. *Let $S_n(t)$ ($n \geq 2$) be as above. We set $S(t) = \sum_{n=2}^{\infty} S_n(t)$. Then we have*

$$\begin{aligned} \int_0^1 S(t)dt = \sum_{n=2}^{\infty} \frac{\zeta(n)}{n}(z^n + (x+y+z)^n \\ -(x+z)^n - (y+z)^n). \end{aligned} \tag{16}$$

Hence we obtain

$$\begin{aligned} \int_0^1 S_n(t)dt = \frac{\zeta(n)}{n}(z^n + (x+y+z)^n \\ -(x+z)^n - (y+z)^n) \end{aligned} \tag{17}$$

for each $n \geq 2$.

Remark. The right-hand side of (16) converges if $|x|$, $|y|$, $|z|$ are sufficiently small.

3. Proof of Theorem 1

Taking the change of integration variable $u = tv$ in (15), we find that each $S_n(t)$ is holomorphic at $t = 0$. Let $s_{n,j}$ be the coefficient of t^j in the power series expansion of S_n in t:

$$S_n(t) = \sum_{j=0}^{\infty} s_{n,j} t^j.$$

By the definition, we have

$$s_{2,j} = \frac{xy}{j+1} \tag{18}$$

and

$$s_{3,j} = (x+y+z)xy \sum_{m=1}^{j+1} \frac{1}{(j+1)m} - \frac{(x+y)xy}{(j+1)^2}. \tag{19}$$

Recurrence formula (15) for S_n ($n \geq 4$) yields the following:

$$\begin{aligned} s_{n,j} = \frac{1}{j+1} \Bigg\{ (x+y+z) \sum_{k=0}^{j} s_{n-1,k} \\ -(x+y)s_{n-1,j} - \sum_{m=2}^{n-2} \sum_{k=0}^{j-1} s_{m,k} s_{n-m,j-1-k} \Bigg\}. \end{aligned} \tag{20}$$

for $n = 2, 3, 4, \ldots$. Hence $S(t) = \sum_{n=2}^{\infty} S_n(t)$ is well-defined as a formal power series in (t, x, y, z) and so is $\int_0^t S(u)du$. By the construction,

$$w = \exp\left(\int_0^t S(u)du\right) \tag{21}$$

satisfies (1) as a formal power series in t and $w|_{t=0} = 1$. Uniqueness of formal power series solution satisfying this condition implies

$$F(x, y, 1-z; t) = \exp\left(\int_0^t S(u)du\right). \tag{22}$$

Taking the limit $t \to 1$ from the left, we find that

$$\int_0^1 S(u)du \tag{23}$$

exists if $|x|$, $|y|$, $|z|$ are sufficiently small and it can be analytically continued to the set

$$\{(x, y, z) \in \mathbb{C}_x \times \mathbb{C}_y \times \mathbb{C}_z | z \notin \mathbb{N}, x+y+z \notin \mathbb{N}\}.$$

Next we consider the following differential operator:

$$\begin{aligned} T = \frac{1-z}{(1-x-z)(1-y-z)} \times \\ \left((1-t)\frac{d}{dt} + 1 - x - y - z\right). \end{aligned} \tag{24}$$

As is well-known, $TF(x, y, 1-z; t) = F(x, y, 2-z; t)$ holds (see [1]) and hence

$$\begin{aligned} T\exp\left(\int_0^t S(x, y, z; u)du\right) \\ = \exp\left(\int_0^t S(x, y, z-1; u)du\right). \end{aligned} \tag{25}$$

On the other hand, the left-hand side of (25) becomes

$$\begin{aligned} \frac{1-z}{(1-x-z)(1-y-z)}((1-t)S(x, y, z; t) \\ +(1-x-y-z))\exp\left(\int_0^t S(x, y, z; u)du\right). \end{aligned} \tag{26}$$

Taking the limit $t \to 1$, we have

$$\begin{aligned} \int_0^1 S(x, y, z; u)du = \int_0^1 S(x, y, z-1; u)du \\ + \log\frac{(1-z-x)(1-y-z)}{(1-z)(1-x-y-z)}. \end{aligned} \tag{27}$$

Therefore we obtain

$$\begin{aligned}\int_0^1 S(x,y,z;u)du &= \int_0^1 S(x,y,z-k;u)du \\ &+ \sum_{j=1}^{k} \log \frac{(j-z-x)(j-y-z)}{(j-z)(j-x-y-z)}.\end{aligned} \tag{28}$$

Since we know

$$\lim_{k\to\infty} F(x,y,1-z+k;1) = 1, \tag{29}$$

we can conclude that

$$\lim_{k\to\infty} \int_0^1 S(x,y,z-k;u)du = 0. \tag{30}$$

Thus we have

$$\begin{aligned}&\int_0^1 S(x,y,z;u)du \\ &= \sum_{j=1}^{\infty} \log \frac{(j-z-x)(j-y-z)}{(j-z)(j-x-y-z)}.\end{aligned} \tag{31}$$

The right-hand side of (31) is expanded easily in a power series of x, y and z:

$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{j^n n}(z^n + (x+y+z)^n \\ -(x+z)^n - (y+z)^n).\end{aligned} \tag{32}$$

This power series converges locally uniformly if $|x|$, $|y|$, $|z|$ are sufficiently small. Hence we may change the order of summation. Note that the summand vanishes if $n = 1$. Thus we obtain

$$\begin{aligned}\sum_{n=2}^{\infty}\sum_{j=1}^{\infty} \frac{1}{j^n n}(z^n + (x+y+z)^n \\ -(x+z)^n - (y+z)^n),\end{aligned} \tag{33}$$

or equivalently,

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{\zeta(n)}{n}(z^n + (x+y+z)^n \\ -(x+z)^n - (y+z)^n).\end{aligned} \tag{34}$$

This completes the proof of the theorem.

References

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