# ２体ディラック方程式と原点波動関数*

伊藤　仁之†

# Two-Body Dirac Equation and Its Wave Function at the Origin

Hitoshi ITO ‡

A relativistic bound state equation is proposed for the Dirac particles interacting through an Abelian gauge field. It reduces to the (one body)Dirac equation in the infinite limit of one of the masses and has the PCT invariance, which assures existence of the anti-bound-state with the same mass. This symmetry is a consequence of a modification of the Stückelberg-Feynman boundary condition for propagation of the negative-energy states, by which some effect of the crossed diagram is taken in the lowest ladder equation. This modification can be corrected back by adding counter correction terms in the interaction kernel when the coupling is weak and the perturbative calculations work well. The interaction kernel is systematically constructed by diagonalizing the Hamiltonian of the underlying field theory. The equation can be used for the quark model phenomenology. Its wave function at the origin(WFO) for the psuedoscalar state becomes finite. Some comments are mentioned for the application in the heavy quark effective theory.[1]

**Key words;** two-body equation, Dirac equation, relativistic dynamics, heavy quark, PCT invariance.

## 1 Introduction

One of the unsatisfactory nature of the Bethe-Salpeter equation for the bound state is that it does not reduce to the Dirac equation in the infinite limit of the one of the constituent mass, even when the interaction is assumed to be instantaneous[2]. We have to sum up all the crossed diagrams to recover the Dirac equation[3], which is impossible for the finite masses. The reducibility to the Dirac equation is essential in applications to the heavy flavored mesons or the other atom-like systems.

Historically, the relativistic single-time equation for the two-body system preceded the BS equation. Soon after the discovery of the Dirac equation, Breit proposed the equation of the form[4]

$$\{H_1(\boldsymbol{p}_1) + H_2(\boldsymbol{p}_2) + V\}\psi = E\psi, \qquad (1)$$

where $H_i$ is the Dirac Hamiltonian

$$H_i(\boldsymbol{p}_i) = \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_i + m_i\beta_i$$

and $V$ is a local potential. The Breit equation reduces to the Dirac equation in the limit mentioned above but does not have the "$E$-parity symmetry", by which we mean there is symmetric negative eigenvalue $E$ for each positive one, which is interpreted as the bound state of the antiparticles.

On the other hand, in the instantaneous BS equation we have an extra projection factor

$$\Lambda_{++} - \Lambda_{--} \qquad (2)$$

in front of the potential $V$, which consists of the energy-projection operators

$$\Lambda_{\varepsilon\eta}(\boldsymbol{p}_1, \boldsymbol{p}_2) = \Lambda^1_\varepsilon(\boldsymbol{p}_1)\Lambda^2_\eta(\boldsymbol{p}_2), \qquad \varepsilon, \eta = + \text{ or } -, \qquad (3)$$

where

$$\Lambda^i_\varepsilon(\boldsymbol{p}_i) = \{E_i(\boldsymbol{p}_i) + \varepsilon H_i(\boldsymbol{p}_i)\}/2E_i(\boldsymbol{p}_i),$$

$$E_i(\boldsymbol{p}_i) = (\boldsymbol{p}_i^2 + m_i^2)^{1/2}.$$

$E$-parity symmetry of this equation is a result of the $PCT$ invariance: The Dirac Hamiltonian is odd under the $PCT$ transformation and the interaction Hamiltonian is assumed to be invariant under it. We therefore see that the symmetry is assured by a projection factor (2). We must introduce a similar factor in any attempt at the construction of an improved two-body equation.

The factor (2) comes from the Stückelberg-Feynman boundary condition for the propagation of the negative-energy state[5]. We will construct the equation for the unequal-mass constituents by imposing similar boundary condition. Before presenting it, we restrict the framework of consideration. We assume Abelian gauge field interacting with the Dirac particles. We will work in the rest frame of the bound system($P = (E, \boldsymbol{0})$), since we are looking for the non-covariant approximation of the low-energy

*平成 10 年 6 月 30 日受理
†数学物理学科
‡E-mail address: itoh@phys.kindai.ac.jp

dynamics. $\boldsymbol{p}$ and $\boldsymbol{x}$, in the following, are the relative momentum and coordinate, respectively, in this frame.

## 2 The equation for unequal-mass constituents

By assuming a modified two-body propagator, we deduce a new single-time equation (Two-body Dirac equation) for the unequal-mass constituents, for which we assume that the mass $M(=m_1)$ is larger than $m(=m_2)$.

### 2.1 Two-body Dirac equation

The BS equation is based essentially on the perturbation theory and consists of the mutually non-interacting propagators and the interaction kernel. This separation may not be a good approximation for the bound state. Then, let us consider to modify this "free" two-body propagator.

We impose the boundary condition that the negative-energy states propagate backward in time. If we keep individuality of the constituents and use the usual Feynman propagator in the instantaneous BS equation, the equation (1) modified by the factor (2) results. However, we can choose the other possibility in which we incorporate the idea that the bound two bodies should be treated as a quantum-mechanical unity.[1] Since $M$ is larger than $m$, the free part of the Hamiltonian has the same sign as that of the particle 1 in the CM system. Let us then modify the boundary condition as follows: *A bound two-particle state propagates backward in time if their net energy is negative.* By assuming it, we have the propagator

$$S_2(P,p) = \sum_{\varepsilon\eta} \frac{1}{\lambda_1 E - p_0 - H_1(-\boldsymbol{p}) + i\varepsilon\delta} \times \frac{1}{\lambda_2 E + p_0 - H_2(\boldsymbol{p}) + i\varepsilon\delta} \Lambda_{\varepsilon\eta}\gamma_0^1\gamma_0^2, \quad (4)$$

where $\lambda_1 + \lambda_2 = 1$ and the limit $\delta \to +0$ is assumed.

By assuming (4) we obtain the single-time equation. The pseudo-4-dimensional form of it, in the momentum space, is

[1] We can establish the concept of individuality in the quantum mechanics only through observation, which brings about a subtle point to the bound system: To detect an individual one in the bound constituents, we need to separate them by applying the 3rd interaction, which inevitably destroys the original state. So there is no *a priori* reason to apply the free propagator individually to each constituent. Note that "the free propagator" itself *for the bound state* is merely a convention of approximation.

$$\psi(p) = iS_2(P,p)\int V(\boldsymbol{p},\boldsymbol{q})\psi(q)d^4q/(2\pi)^4, \quad (5)$$

where $V(\boldsymbol{p},\boldsymbol{q})$ represents the interaction, which does not depend on the relative energies but is not necessarily an instanteneous local potential. After integrating out the redundant degree of the freedom, (5) becomes

$$\{H_1(-\boldsymbol{p}) + H_2(\boldsymbol{p}) + \sum_{\varepsilon\eta} \varepsilon\Lambda_{\varepsilon\eta}V\}\psi = E\psi \ , \quad (6)$$

which reduces to the Dirac equation in the infinite limit of $M$. It is easy to see the $E$-parity symmetry of this equation.

If we would apply our equation to the scattering state, we shall have, from the time-dependent equation, the conserved probability density

$$\rho(\boldsymbol{x},t) = \psi(\boldsymbol{x},t)^\dagger \sum_{\varepsilon\eta} \varepsilon\Lambda_{\varepsilon\eta}\psi(\boldsymbol{x},t), \quad (7)$$

which is in accord with the boundary condition that the negative- energy state propagates backward in time carrying the negative probability density[6]. However, it does not necessarily provide the normalization condition for the bound state: For the scattering processes, the projected wave function $\Lambda_{\varepsilon\eta}\psi$ corresponds to the physical state of the (free)particles with the positive or negative energy $E$. And the above interpretation of the probability current actually says that *the state with the negative $E$ is carrying the negative probability density.* However, for the bound state with the positive eigenvalue $E$, $\Lambda_{--}\psi$ or $\Lambda_{-+}\psi$ is merely *a negative-energy component in the representation in which the energy of the free particle is diagonal.*

Taking the above consideration into account, we restore the probability interpretation of the wave function and normalize it by assuming the probability density

$$\rho(\boldsymbol{x}) = \psi(\boldsymbol{x},t)^\dagger\psi(\boldsymbol{x},t), \quad (8)$$

for the stationary state. Observables except the Hamiltonian are self-adjoint under this metric:

$$(\phi, \hat{O}\psi) = (\hat{O}\phi, \psi). \quad (9)$$

The Hamiltonian is the operator ruling the time development of the system and its interaction part is modified by the factor $\sum_{\varepsilon\eta} \varepsilon\Lambda_{\varepsilon\eta}$. Though it is not a self-adjoint operator, its eigenvalue is proved to be real if the inner product (13) below exists.

## 2.2 Green's function and the vertex equation

We define the Green's function $G$ for (6) by the operator equation

$$\{E - H_1 - H_2 - \sum_{\varepsilon\eta} \varepsilon\Lambda_{\varepsilon\eta} V\}G = \sum_{\varepsilon\eta} \varepsilon\Lambda_{\varepsilon\eta}, \quad (10)$$

and

$$G\{E - H_1 - H_2 - V\sum_{\varepsilon\eta} \varepsilon\Lambda_{\varepsilon\eta}\} = \sum_{\varepsilon\eta} \varepsilon\Lambda_{\varepsilon\eta}. \quad (11)$$

In the momentum representation, it can be written, by using the eigen-function $\chi_n(\boldsymbol{p})$ of (6), as

$$G(\boldsymbol{p}, \boldsymbol{p}') = \sum_n \frac{1}{N_n(E - E_n)} \chi_n(\boldsymbol{p})\chi_n(\boldsymbol{p}')^\dagger + \text{continuum}, \quad (12)$$

where $E_n$ is an eigenvalue and $N_n$ is a normalization factor defined by

$$N_n = (\chi_n, \sum_{\varepsilon\eta} \varepsilon\Lambda_{\varepsilon\eta}\chi_n). \quad (13)$$

When one of the constituents(labeled with 2) belongs to the same class of the antiparticle of the other, there can be an annihilation process through the weak interaction for which the unamputated-decay-vertex $\Phi$ is given by

$$\Phi = C\gamma G, \quad (14)$$

where $\gamma$ is the lowest vertex and $C$ is the charge-conjugation matrix of the particle 2.

$\Phi$ satisfies the vertex equation

$$\Phi(E - H_1 - H_2 - V\sum_{\varepsilon\eta} \varepsilon\Lambda_{\varepsilon\eta}) = C\gamma \sum_{\varepsilon\eta} \varepsilon\Lambda_{\varepsilon\eta} \quad (15)$$

and the amputated vertex is

$$\Gamma = \gamma + C\Phi V. \quad (16)$$

We can determine the renormalization constant $Z_1$ for the wave function at the origin(WFO) from (15) and (16), if we need it[7].

## 2.3 Interaction Hamiltonian

We have, so far, not specified the interaction Hamiltonian (quasipotential). In this section, we investigate it for the one-(Abelian)gauge-boson exchange in the Coulomb gauge as an example. For the instantaneous part of the interaction, $V$ is obvious. For the remaining part, we can specify the quasipotential in a clear way from the background field theory. We have already shown, for the Salpeter equation, that the quasipotential from the one-boson exchange is given through the diagonalization of Fukuda, Sawada, Taketani[8] and Okubo[9] (FSTO)[10]. We also apply this method to the present equation.

We introduce the generalized Fock subspace of the free constituents, the bases of which are denoted by $|\varepsilon, \eta, \boldsymbol{p}\rangle$, where $\varepsilon$ and $\eta$ are the signs of the energies of the particles 1 and 2 respectively.[2] We then diagonalize the Hamiltonian in the Schrödinger picture by using the FSTO method. The second-order boson-exchange potential in this subspace is given by

$$\begin{aligned}\langle \varepsilon, \eta, \boldsymbol{p}|V(1\mathrm{b})|\varepsilon', \eta', \boldsymbol{p}'\rangle = &\\ \frac{g^2}{(2\pi)^3}\sum_{ij} \alpha_{1i}(\delta_{ij} - \frac{1}{\boldsymbol{q}^2}q_i q_j)\alpha_{2j} &\\ \times\frac{1}{2}\Big[\frac{1}{\boldsymbol{q}^2 - \{\varepsilon E_1(p) - \varepsilon' E_1(p')\}^2} &\\ + \frac{1}{\boldsymbol{q}^2 - \{\eta E_2(p) - \eta' E_2(p')\}^2}\Big], &\quad (17)\end{aligned}$$

where $\boldsymbol{q} = \boldsymbol{p} - \boldsymbol{p}'$. The retardation effects are included in this equation.

## 2.4 On the equal-mass limit

The unequal-mass equation (6) is well applied to the system with $M \gg m$. For $M \simeq m$, the reason justifiing (4) becomes obscure and we will have two different equations in the limit $M = m$. Though we concern ourselves in the case $M > m$, it is meaningful to investigate the degree of ambiguity near the equal mass limit. For $M = m$, the projection factor in front of the interaction term of (6) includes a part which violates the exchange symmetry: It is shown that

$$\Lambda^{(V)} = \Lambda_{+-} - \Lambda_{-+} \quad (18)$$

and the Heisenberg's exchange operator

$$P_H = \frac{1}{4}(1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)(1 + \boldsymbol{\rho}_1 \cdot \boldsymbol{\rho}_2)P_M \quad (19)$$

anticommute, where the operator $P_M$ exchanges the momenta(or coordinates). $\Lambda^{(V)}$ violates the symmetry since the remaining part of the Hamiltonian and $P_H$ commute.

For equal-mass limit, we have two equations. One is the equation (6) with $M = m$ and another is obtained by assigning a minus sign in front of $\Lambda^{(V)}$,

[2] There was some error concerning the Fock space in Ref.[10]. Namely, we employed the usual(positive-energy) Fock space and reinterpreted the matrix elements of the interaction Hamiltonian including the negative-energy indices as the ones in this space according to the hole theory. It, however, brings the procedure into confusion, since we have revived the negative energy in Eq.(6). However, the error is only conceptual for the Salpeter equation which has only projection factors $\Lambda_{++}$ and $\Lambda_{--}$. The result of Ref.[10] is correct.

which is the equal-mass limit of the equation with $m > M$. It is the conjugate equation of (6) in the sense that (6) is converted into it by the transformation $P_H$. It is easy to show that these equations have the common eigenvalue spectrum: If an eigenfunction $\chi_n$ of (6) belongs to some eigenvalue $E_n$, $P_H\chi_n$ is the solution of the conjugate equation with the same eigenvalue. However, $\chi_n$ does not have the definite $P_H$-parity.

For equal masses, it is reasonable to use the Salpeter equation which includes the projection factor (2). The eigenvalue of this equation is different from the above $E_n$. The difference is, however, small since it is of the 4th order in the symmetry-breaking part of the Hamiltonian.

# 3 An effect of the modified propagator

We have modified the two-body propagator as (4) in the CM system. For general momentum, the modified propagator of the lighter particle 2 is written as

$$\tilde{S}_F^2(p;\varepsilon) \equiv \sum_\eta \tilde{S}_F^2(p;\varepsilon,\eta) = \sum_\eta \frac{1}{p_0 - \eta E_2(\boldsymbol{p}) + i\varepsilon\delta}\Lambda_\eta^2(\boldsymbol{p})\gamma_0^2, \tag{20}$$

where $\varepsilon$ is the signature factor in the definition of the propagator of the heavier particle 1:

$$S_F^1(p) \equiv \sum_\varepsilon S_F^1(p;\varepsilon) = \sum_\varepsilon \frac{1}{p_0 - \varepsilon E_1(\boldsymbol{p}) + i\varepsilon\delta}\Lambda_\varepsilon^1(\boldsymbol{p})\gamma_0^1, \tag{21}$$

where $p = (p_0, \boldsymbol{p})$. We have modified the $(\varepsilon,\eta) = (+,-)$ and $(-,+)$ parts in (20).

Now, we justify, in the framework of the perturbation theory, the above modification in the large $M$ limit by considering the scattering of the mass-shell particles. We estimate the contribution of the modified parts to the intermediate state of the 4th order ladder(box) diagram and show that it can be interpreted as the substitute for the ordinary Feynman propagatoer in the crossed diagram. We, further, discuss the counter correction needed for the perturbation theory.

## 3.1 Effective inclusion of a crossed diagram

We take up the 4th-order amplitude with the Coulomb potential which dominates the interaction in the large $M$ limit. The contribution of the intermediate state with $(\varepsilon,\eta) = (-,+)$ is much smaller $(O(1/M))$ than that of the state with $(\varepsilon,\eta) = (+,-)$. Now, let us compare the two amplitudes restricted to this intermediate state: the ladder amplitude($\tilde{M}_l(+,-)$) with the propagators defined by (20) and (21) and the amplitude from the crossed diagram($M_c(+,-)$) with the Feynman propagators.

$\tilde{M}_l(+,-)$ is given by

$$\tilde{M}_l(+,-) = \frac{i\alpha^2}{\pi^2}\int d^4q \frac{1}{\boldsymbol{q}^2(\boldsymbol{q}-\boldsymbol{p}_2+\boldsymbol{p}_2')^2} \times\gamma_0^1 S_F^1(p_1+q;+)\gamma_0^1\gamma_0^2\tilde{S}_F^2(p_2-q;+-)\gamma_0^2, \tag{22}$$

where $\alpha = g^2/4\pi$ and $p_1$, $p_2$ and $p_1'$, $p_2'$ are the initial and final momenta respectively.

If we take the large $M$ limit in (22) and retain only the leading term, it becomes

$$\tilde{M}_l(+,-) \approx \frac{\alpha^2}{\pi}(1+\gamma_0^1)\int d^3q \times\frac{1}{\boldsymbol{q}^2(\boldsymbol{q}-\boldsymbol{p}_2+\boldsymbol{p}_2')^2}\frac{\gamma_0^2\Lambda_-^2(\boldsymbol{p}_2-\boldsymbol{q})}{E_2(\boldsymbol{p}_2)+E_2(\boldsymbol{p}_2-\boldsymbol{q})}. \tag{23}$$

We compare this with the same approximation to the $M_c(+,-)$ which is given by

$$M_c(+,-) \approx \frac{\alpha^2}{\pi}(1+\gamma_0^1)\int d^3q \times\frac{1}{\boldsymbol{q}^2(\boldsymbol{q}+\boldsymbol{p}_2-\boldsymbol{p}_2')^2}\frac{\gamma_0^2\Lambda_-^2(\boldsymbol{p}_2'-\boldsymbol{q})}{E_2(\boldsymbol{p}_2')+E_2(\boldsymbol{p}_2'-\boldsymbol{q})}. \tag{24}$$

The equation (23) approximates (24) for the momenta $\boldsymbol{p}_2' \approx \boldsymbol{p}_2$. Since the forward scattering is dominant for the Coulomb force, we can regard the modification of the propagator in (4) or (20) as the substitute for including a part of the crossed diagram.

The (pseudo)potential model is effective in the strong coupling theory, for which the effective inclusion of the crossed diagram is desirable.

## 3.2 The counter correction

For the weakly binding system, the two-body Dirac equation (6) offers a good basis for the perturbation theory, becuase we can start with the (one-body)Dirac equation for the unperturbed system. Then, we can develop the systematic expansion in the power series of $1/M$ and the coupling constant.

We should correct back the modification of the propagator in this series of the approximation, by adding the counter correction term to the propagator of the light particle in the calculation of the crossed diagram, for example. A similar correction is needed wherever, in a diagram, the ladder iteration of the modified parts of the propagator (4) may contribute. The counter correction term is given by taking the difference of the modified propagator defined by (20) and (21) and the Feynman propagator: that is

$$2i\varepsilon\pi\delta(p_0 - \eta E_2(p))\delta_{\eta,-\varepsilon}$$

for each innerline of the particle 2 with $(\varepsilon, \eta) = (+,-)$ or $(-,+)$, where $p$ is the momentum flowing on the line.

# 4 Application to the Quark Model of Mesons

We can use our equation as the basic equation of the quark model, if we add the phenomenological confining potential. We mention some related properties in this section.

## 4.1 WFO of the $^1S_0$ state

When the equation is applied to the system in which the pair annihilation of the constituents can occur, an important physical quantity is the wave function at the origin(WFO). For example, the decay amplitude of the pseudo-scalar $Q\bar{q}$ meson via a weak boson is proportional to the average WFO $\mathrm{Tr}\{\gamma_5\gamma_0\Psi(0)\}$, where $\Psi(0)$ is the charge conjugated(with respect to the particle $2(\bar{q})$) WFO. We investigate, in the Appendix, the asymptotic behavior of the momentum-space wave function by using the method given in Ref.[7]. We assume instantaneous exchange of a gauge boson[3] . The average WFO thus obtained is finite. This result is consistent with a consideration on the structure of the covariant bound state.

There are many "two-body Dirac equation" proposed. An interesting one from the point of view of the present paper is the one by Mandelzweig and Wallace[11]. They intended to include the effects of the higher-order interaction (the crossed Feynman diagram) and got an equation which has the proper one-body limit and the $E$-parity symmetry. An important difference from our equation is in the average WFO considered above. It is divergent in their equation if the transverse part of the gauge-boson exchange is included[12].

## 4.2 On application to the heavy quark effective theory

One of the research fields in which we can utilize the two-body Dirac equation is the physics of the heavy flavored mesons. The recent trends in this field are led by the heavy quark effective theory(HQET)[13]. The $Q\bar{q}$ system is well described by using the two-body Dirac equation with some phenomenological (pseudo)potential. If we take the heavy quark limit($M \to \infty$) of the quark $Q$, the equation itself becomes the one-body Dirac equation, which is the underlying basic equation of HQET.

There are, however, divergent(as $M \to \infty$) portions in the perturbative correction to the WFO. For example, the matrix element of the current for the annihilation decay of a pseudoscalar meson is given by

$$< 0|\bar{q}\gamma_\nu\gamma_5 Q|Q\bar{q} >= -C(\mu)\mathrm{tr}\{\gamma_\nu\gamma_5\Psi^R(0)\}, \quad (25)$$

where $\Psi^R(0)$ is the renormalized WFO of the one-body Dirac equation[14]. $C(\mu)$ is determined from the perturbative loop corrections summed up by using the renormalization group equation [15][16].

$$C(\mu) = (\frac{\alpha_s(M)}{\alpha_s(\mu)})^d, \quad d = -6/(33 - 2N_f). \quad (26)$$

One of the subjects to be investigated in HQET is to solve directly the bound state equation. We can use the two-body Dirac equation supplemented the expansion in $1/M$ for this purpose.

## 4.3 Another approach to the heavy quark phenomenology

An alternative way of investigating the heavy quark systems is due to direct application of the two-body Dirac equation. The binding interaction, in this formulation, consists of the Coulomb and the confining potentials. We further add the gluonic correction term to the potential, for which we should attach the high momentum cutoff in the case of the heavy quarkonia[17]. [4]

The heavy mass($M$) dependence is properly described by (6). However, for the annihilation decays, $M$ dependence of the amplitude comes also from the high momentum of the loop corrections, which is represented by a similar factor to $C(\mu)$ above. $C(\mu)$ depends on the cutoff to the gluonic correction. If we take the cutoff of the QCD scale, we should include all the corrections from the high momenta, except for the Coulomb contribution to the vertex correction. By subtracting its effect, we get $d = 10/(33 - 2N_f)$ for the exponent. [5]

[3] The analysis in the Appendix cannot be applied to the retarded interaction.

[4] We note that we may include the gluonic correction from the momentum region larger than $M$ in the equation, since the light quark is not sensitive to the inner most interaction and we may avoid the singular behavior of the mass spectrum.

[5] The correction was calculated for $< 0|\bar{q}\gamma_0\gamma_5 Q|Q\bar{q} >$. And

# 5 Summary and discussion

By assuming a proper boundary condition for the two-body propagation in the negative-energy state, we have modified the propagator and proposed a new bound-state equation of the unequal-mass constituents for Abelian gauge interactions. The effect of a part of the crossed diagram are included in the ladder model by this modification of the propagator.

The equation has the symmetrical energy eigenvalues $E_n$ and $-E_n$ and reduces to the (one-body)Dirac equation in the infinite limit of one of the constituents masses. Secondly, we have discussed the normalization of the wave function and pointed out that the positive-definite probability density should be assumed. We can consistently calculate the observables of the bound state by assuming this normalization.

The interaction Hamiltonian of the equation is constructed by diagonalizing the field theoretic Hamiltonian in the generalized Fock subspace of the two particles. Relativistic effects such as the retardation are taken into account in systematic way in the framework of the perturbation theory.

We can use this two-body equation as the foundation of the perturbation theory of the weak-coupling theory. In this case, we should correct back the modification of the propagator.

We have further investigated the WFO's of the proposed equation in some detail for the instantaneous interaction and shown that the average WFO in the ${}^1S_\bullet$ state which determines the leptonic decay rate of a pseudo-scalar meson is finite, which is in accord with an expectation from the structure of the covariant bound state.

One of the field for which we may utilize the two-body Dirac equation is the heavy quark effective theory. There are various possibilities of choosing the framework of the approximation according to the treatment for the high-momentum interaction. We have briefly discussed the correction factor to the leptonic decay width caused by the high momenta. The equation affords a good foundation of the heavy quark effective theory.

Finally, we discuss some feature of the spectrum of the equation (6). There are physical eigenvalues $E_n$ which reduce to $M+m$ in the weak coupling limit and the corresponding negative ones which are interpreted as the bound states of the antiparticles. Besides those, there may be a series of eigenvalues which reduce to $M-m$ in the weak-coupling limit and its negative counter part, which are unphysical. Fortunately, we can identify and discard these unphysical spectra by inspecting the weak-coupling limit.

we have assumed the counter correction which compensates the modification of the two-body propagator, since it cannot be regarded as a substitute for crossed diagram in QCD. If we do not include the counter correction we shall get $d = 2/(33 - 2N_f)$[18].


# Acknowledgements

The author would like to thank Professor G.A. Kozlov for useful communication on the two-time Green's function.


# Appendix

We examine the asymptotic ($p \to \infty$) behavior of the momentum-space wave function and show that the average WFO $\mathrm{Tr}\{\gamma_5\gamma_0\psi(0)\}/\sqrt{2}$ is finite.[6]

There are 4 partial amplitudes $h_{\varepsilon\eta}(p)$ in the ${}^1S_\bullet$ state, with which the wave function is expanded as

$$\chi(p) = \sum_{\varepsilon\eta}\sum_{r} c_r u_\varepsilon^r(-p) v_\eta^{-r}(p) h_{\varepsilon\eta}(p) \left(\frac{1}{16\pi E_1 E_2}\right)^{1/2}, \tag{27}$$

where $c_{1/2} = -c_{-1/2} = 1/\sqrt{2}$ and the spinors $u$ and $v$, for the particle 1 and 2 respectively, are defined in [7].

The average WFO for the annihilation decay through the axial-vector current is given by

$$\begin{aligned}\frac{1}{\sqrt{2}}\mathrm{Tr}\{\gamma_5\gamma_0\psi(0)\} &= \frac{1}{\sqrt{8}\pi}\int\left(\frac{1}{E_1E_2}\right)^{1/2} \\ &\times \sum_{\varepsilon\eta}\{\sqrt{(E_1+\varepsilon M)(E_2+\eta m)} \\ &-\varepsilon\eta\sqrt{(E_1-\varepsilon M)(E_2-\eta m)}\} h_{\varepsilon\eta}(p) p^2 dp.\end{aligned} \tag{28}$$

We first assume the Coulomb potential. The partial-wave equation for the ${}^1S_\bullet$ state is given by

$$\begin{aligned}\{E - \varepsilon E_1(p) - \eta E_2(p)\} h_{\varepsilon\eta}(p) &= -\varepsilon\frac{\alpha}{4\pi}\sum_{\varepsilon'\eta'}\int dq \\ &\times \frac{q}{p}\left[\frac{1}{E_1(p)E_1(q)E_2(p)E_2(q)}\right]^{1/2} \\ &\times [\{A^1_{\varepsilon\varepsilon'}A^2_{\eta\eta'} + \varepsilon\varepsilon'\eta\eta' A^1_{-\varepsilon-\varepsilon'}A^2_{-\eta-\eta'}\}Q_0(z) \\ &+ \{\varepsilon\varepsilon' A^1_{-\varepsilon-\varepsilon'}A^2_{\eta\eta'} + \eta\eta' A^1_{\varepsilon\varepsilon'}A^2_{-\eta-\eta'}\}Q_1(z)] h_{\varepsilon'\eta'}(q),\end{aligned} \tag{29}$$

where $z = (p^2+q^2)/2pq$ and $Q_\ell(z)$ is the Legendre's function. $A^i_{\varepsilon\varepsilon'}$ is defined by

$$A^i_{\varepsilon\varepsilon'} = \sqrt{(E_i(p)+\varepsilon m_i)(E_i(q)+\varepsilon' m_i)}.$$

[6] See Ref.[19] and references therein, for the Salpeter equation.

The asymptotic behavior of the wave function is determined from the integral region near the infinity. We then expand the both sides of (29) into the series of $1/p$ and $1/q$. We assume the power behavior of the wave function for large $p$. The independent amplitudes are chosen to be $h_A(p) \equiv \sum_\varepsilon h_{\varepsilon\varepsilon}(p)$, $h_B(p) \equiv \sum_\varepsilon \varepsilon h_{\varepsilon\varepsilon}(p)$, $h_C(p) \equiv \sum_\varepsilon h_{\varepsilon-\varepsilon}(p)$, and $h_D(p) \equiv \sum_\varepsilon \varepsilon h_{\varepsilon-\varepsilon}(p)$, which are expanded, in the high-momentum region, in power series of $1/p$:

$$h_X(p) = \sum_n C_X^n p^{-\beta_X - 2n - 1}$$

Integrals on the right-hand side can be done if we neglect infrared- divergent terms which are irrelevant to the leading asymptotic behavior. Now, we can determine the asymptotic indices $\beta_X$'s from consistency[7]: We get, for $h_A$ and $h_B$,

$$\begin{aligned} 2C_A^0 p^{-\beta_A} - E C_B^0 p^{-\beta_B - 1} = \\ \frac{\alpha}{\pi} C_A^0 \frac{\pi}{1-\beta_A} \cot(\frac{\pi}{2}\beta_A) p^{-\beta_A} \end{aligned} \tag{30}$$

and

$$\begin{aligned} 2C_B^0 p^{-\beta_B} - E C_A^0 p^{-\beta_A - 1} = \\ \frac{\alpha}{\pi} C_B^0 \frac{\pi(1-\beta_B)}{\beta_B(2-\beta_B)} \tan(\frac{\pi}{2}\beta_B) p^{-\beta_B}, \end{aligned} \tag{31}$$

where the terms of the higher power in $1/p$ are neglected. If we neglect the second term in the left-hand sides of (30), we find $\beta_A$ in the range $1 < \beta_A < 2$ [7] and get $\beta_B = \beta_A + 1$ from (31). We obtain another series by neglecting the second term in (31). For this, $\beta_B$ is found to be in the range $2 < \beta_0 < \beta_B < 3$, where the lower bound $\beta_0$ corresponds to the upper bound $4/\pi$ of $\alpha$ above which the index $\beta_A$ from (30) becomes complex. $\beta_A$ of the second series is given by $\beta_A = \beta_B + 1$.

The asymptotic amplitudes $h_C$ and $h_D$ are determined dependently on $h_A$ and $h_B$. We get, for the minimum indices

$$\beta_C = \min(\beta_A + 2, \beta_B + 1) \tag{32}$$

$$\beta_D = \beta_A + 1. \tag{33}$$

We see that the average WFO (28) is finite, because

$$\beta_B > 1 \text{ and } \beta_C > 2$$

hold for the asymptotic amplitudes. This conclusion is valid even if the instantaneous exchange(transverse part) of the gauge boson is added.

[7] See Ref.[7] for the details.

# References


[1] A preliminary report was presented in the workshop "Fundamental Problems in the Elementary Particle Theory" held at Nihon University in March 1995 and 1996 YITP Workshop "Recent Developments in QCD and Hadron Physics, December 1996. The outline of the final form was reported in the 7th International Conference on Hadron Spectroscopy held at the Brookhaven National Laboratory, August 1997.

[2] E. E. Salpeter. *Phys. Rev.***87**,*328*, 1952. There is another limiting procedure in which the ladder BS equation becomes the Dirac equation:

C. Hayashi and Y. Munakata. *Prog. Theor. Phys.***7**,*481*, 1952.

However, we take the Salpeter's approximation first, since we are looking for the finite-mass single-time equation which reduces to the Dirac equation.

[3] S.J. Brodsky. *Proc. of Brandeis University Summer Institute in Theoretical Physics, 1969,* **1**, *p.91.* Gordon and Breach, 1971.

[4] G.Breit. *Phys. Rev.***34**,*553*, 1929; **36**, *383*, 1930.

[5] E. C. G. Stückelberg. *Helv. Phys. Acta***14**,*32L,588*, 1941.

R. P. Feynman. *Phys. Rev.***76**,*749*, 1949.

[6] See, for example, I.J.R. Aitchison and J.G. Hey A. *Gauge Theories in Particle Physics, p.56.* Adam Hilger, Bristol, 1982.

[7] H. Ito. *Prog. Theor. Phys.***67**,*1553*, 1982.

[8] N. Fukuda K. Sawada and M. Taketani. *Prog. Theor. Phys.***12**,*156*, 1954.

[9] S. Okubo. *Prog. Theor. Phys.***12**,*603*, 1954.

[10] H. Ito. *Prog. Theor. Phys.***84**,*94*, 1990. We have assumed larger values for the cutoffs of the gluon contribution and developed the phenomenology of the leptonic decays. The vertex equation defining the renormalization constant $Z_1$ of this paper took in the gluonic correction.

[11] V.B. Mandelzweig and S.J. Wallace. *Phys. Lett.***B197**,*469*, 1987.

S.J. Wallace and V.B. Mandelzweig. *Nucl. Phys.***A503**,*673*, 1989.

[12] P.C. Tiemeijer. *Relativistic Analysis of the Constituent Quark Model for Mesons, p.1.* Universiteit Utrecht, Utrecht, 1993.

[13] See, N. Isgur and M.B. Wise. *Heavy Flavours(eds. A.J. Buras and M. Linder), p.234*. World Scientific, Singapore, 1992.

and A.F. Falk. *The Heavy Quark Expansion of QCD.* The Johns Hopkins University, JUH-TIPAC-96017.

[14] H. Ito. *Prog. Theor. Phys.*78,*978*, 1987; **83**,*1064(Errata)*, 1990; **89**, *763*, 1993.

[15] M.B. Voloshin and M.A. Shifman. *Sov. J. Nucl. Phys.*45,*292*, 1987.

[16] H.D. Politzer and M.B. Wise. *Phys. Lett.*B206,*681*, 1988.

[17] H. Ito. *Prog. Theor. Phys.*71,*868*, 1984. See also Ref.9).

[18] H. Ito. *Proc. of the International Symposium on Extended Object and Bound System, March 1992, Karuizawa, p.131*. World Scientific, Singapore, 1993.

[19] H. Ito. *Prog. Theor. Phys.*80,*874*, 1988.