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A strong duality for separable C^*-algebras

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可分 C^*-環の強双対性

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It is shown that every separable C^*-algebra can be reconstructed from the space of all its irreducible representations on a separable Hilbert space.

Keywords : C^*-algebra, representation, irreducible decomposition, duality

1. Introduction

In [4] Takesaki showed a duality for C^*-algebras such that every separable C^*-algebra is *-isomorphic to the algebra of certain operator fields on the space of all its representations on a separable Hilbert space. It is not only a generalization of Gelfand representation theorem for commutative C^*-algebras to the non-commutative case but also the initiative duality for C^*-algebras regarding the space of representations as a dual object.

Here in this paper, we take the space of only irreducible representations as a dual object for separable C^*-algebras and prove that a strong dual version of Takesaki's duality is still valid. The name of strong duality comes from the theory of group duality.

In [3] Fujimoto accomplished a strong duality for general C^*-algebras using cp-convexity theory and crucial results in [1]. Our method in this paper is quite different. Reducing the argument to Takesaki's duality we prove a strong duality by means of the disintegration theory of representations. So our concern is to formulate and prove a strong duality within the framwork of the representation theory.

2. Operator fields on $\mathrm{Irr}(A{:}H)$

Throughout of this paper A denotes a separable C^*-algebra and H a separable Hilbert space.

Let $\mathrm{Rep}(A{:}H)$ denote the set of all representations of A on H, where a representation of A on H means a *-homomorphism of A into the full bounded operator algebra $\mathfrak{B}(H)$ on H.

For each π in $\mathrm{Rep}(A{:}H)$, let H_π be the essential space of π, closed subspace spanned by $\pi(A)H$, and p_π the projection of H onto H_π.
By the separability of H it follows that $\dim H_\pi = 0, 1, 2, \cdots, \aleph_0$. If K is a closed subspace of H and σ is a representation of A on K, then we identify the representation $\{\sigma, K\}$ with the representation $\{\sigma \oplus O_{K^\perp}, K \oplus K^\perp\}$ belongs to $\mathrm{Rep}(A{:}H)$, where $O_{K^\perp}$ is the zero representation of A on the space of the orthogonal complement of K.

Let $\mathrm{Irr}(A{:}H)$ denote the set of all $\pi \in \mathrm{Rep}(A{:}H)$ such that $\{\pi, H_\pi\}$ is irreducible and let $\mathrm{Irr}_n(A{:}H)$ be the set of all $\pi \in \mathrm{Irr}(A{:}H)$ such that $\dim H_\pi = n$, where $n = 1, 2, \cdots, \aleph_0$. Each $\mathrm{Irr}_n(A{:}H)$ is endowed with the coarest Borel structure which makes $\mathrm{Irr}_n(A{:}H) \ni \pi \mapsto (\pi(a)\xi|\eta)$ a Borel function for every $a \in$ A and every $\xi, \eta \in H_\pi$.

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The set $\mathrm{Irr}(A{:}H) = \bigcup_{n=1}^{\aleph_0} \mathrm{Irr}_n(A{:}H)$ has sum-Borel-structure.

Let $U\pi_1 U^* = \pi_2$ denote the unitary equivalence of $\pi_1, \pi_2 \in \mathrm{Rep}(A{:}H)$ under an intertwiner U.

By the disintegration theory of representations, each non-zero π in $\mathrm{Rep}(A{:}H)$ can be decomposed as follows ;

$$U\{\pi, H_\pi\}U^* = \int_Z^{\oplus} \{\pi(\zeta), H_\pi(\zeta)\}\, d\mu(\zeta)$$

, where μ is a positive measure on a standard Borel space Z and $\pi(\zeta) \in \mathrm{Irr}(A{:}H)$ $\mu - a.e.$

We shall call this decomposition *an irreducible decomposition of* π *over* $\mathrm{Irr}(A{:}H)$.

It is well known that unless A is of type-I this decomposition is not unique.

Definition 1. Let A[Irr($A{:}H$)] denote the set of all $\mathfrak{B}(H)$-valued function x defined on $\mathrm{Irr}(A{:}H)$ satisfies the following conditions (a),(b),(c) and (d) ;

(a) $x(\pi) = p_\pi x(\pi) p_\pi$ for every $\pi \in \mathrm{Irr}(A{:}H)$,

(b) $\mathrm{Irr}(A{:}H) \ni \pi \mapsto (x(\pi)\xi|\eta)$ is a Borel function for every $\xi, \eta \in H$,

(c) $\sup\{\|x(\pi)\| : \pi \in \mathrm{Irr}(A{:}H)\} < +\infty$ and

(d) If $\pi \in \mathrm{Rep}(A{:}H)$ and

$$U_i \pi U_i^* = \int_{Z_i}^{\oplus} \pi_i(\zeta)\, d\mu_i(\zeta) \quad (i = 1, 2)$$

are irreducible decompositions of π over $\mathrm{Irr}(A{:}H)$, then it holds that

$$U_1^* \int_{Z_1}^{\oplus} x(\pi_1(\zeta))\, d\mu_1(\zeta)\, U_1 = U_2^* \int_{Z_2}^{\oplus} x(\pi_2(\zeta))\, d\mu_2(\zeta)\, U_2.$$

For any $x, y \in$ A[Irr($A{:}H$)] and complex numbers α, β, for each $\pi \in \mathrm{Irr}(A{:}H)$ put

$$\begin{aligned}(\alpha x + \beta y)(\pi) &:= \alpha x(\pi) + \beta y(\pi),\\ (x y)(\pi) &:= x(\pi)\, y(\pi),\\ x^*(\pi) &:= x(\pi)^* \textit{ and}\\ \|x\| &:= \sup\{\|x(\pi)\| : \pi \in \mathrm{Irr}(A{:}H)\}.\end{aligned}$$

Then it is easily seen that A[Irr($A{:}H$)] becomes a normed*-algebra with $\|x^* x\| = \|x\|^2$. Furthermore we have

Proposition 1. A[Irr($A{:}H$)] *is a C^*-algebra which contains the *-subalgebra isomorphic to* A.

Proof. We begin with the completeness of A[Irr($A{:}H$)]. Let $\{x_\alpha\}$ be a Cauchy net in A[Irr($A{:}H$)]. Then by the preceding definition it follows that

$$\begin{aligned}\|x_\alpha - x_\beta\| &= \sup\{\|x_\alpha(\sigma) - x_\beta(\sigma)\| : \sigma \in \mathrm{Irr}(A{:}H)\}\\ &\geq \|x_\alpha(\sigma) - x_\beta(\sigma)\| \quad \forall \sigma \in \mathrm{Irr}(A{:}H).\end{aligned}$$

So for each $\sigma \in \mathrm{Irr}(A{:}H)$, $\{x_\alpha(\sigma)\}$ converges uniformly to a certain operator of $\mathfrak{B}(H)$ which we denote by $x(\sigma)$ and the convergence $x_\alpha(\sigma) \to x(\sigma)$ is uniformly on $\mathrm{Irr}(A{:}H)$. It is easily seen that the operator field x, defined on $\mathrm{Irr}(A{:}H)$, satisfies the conditions (a), (b), (c) of Definition 1. Since it holds that

$$\begin{aligned}&\Big\| \int_Z^{\oplus} \{x(\pi(\zeta)) - x_\alpha(\pi(\zeta))\}\, d\mu(\zeta) \Big\|\\ &\leq \mathrm{ess\ sup}\{\|x(\pi(\zeta)) - x_\alpha(\pi(\zeta))\| : \zeta \in Z\}\\ &\leq \sup\{\|x(\sigma) - x_\alpha(\sigma)\| : \sigma \in \mathrm{Irr}(A{:}H)\}\end{aligned}$$

for any irreducible decomposition

$$U\pi U^* = \int_Z^{\oplus} \pi(\zeta)\, d\mu(\zeta)$$

of $\pi \in \mathrm{Rep}(A{:}H)$ over $\mathrm{Irr}(A{:}H)$, it follows under the assumption of (d) that

$$\begin{aligned}&\Big\|U_1^* \int_{Z_1}^{\oplus} x(\pi_1(\zeta))\, d\mu_1(\zeta)\, U_1\\ &- U_2^* \int_{Z_2}^{\oplus} x(\pi_2(\zeta))\, d\mu_2(\zeta)\, U_2\Big\|\\ &\leq 2\sup\{\|x_\alpha(\sigma) - x(\sigma)\| : \sigma \in \mathrm{Irr}(A{:}H)\} \to 0.\end{aligned}$$

This implies that x satisfies the condition (d), so x belongs to A[Irr($A{:}H$)]. Hence A[Irr($A{:}H$)] is complete, so it is a C^*-algebra.

For each $a \in A$, put $\hat{a}(\pi) = \pi(a)$ for every $\pi \in \mathrm{Irr}(A{:}H)$. Then it is obvious that the operator field $\hat{a}$ satisfies the all conditions of Definition 1 and the map. $A \ni a \mapsto \hat{a} \in$ A[Irr($A{:}H$)] is a *-homomorphism. We shall call this map. a generalized Gelfand transform. Since it holds by the plenitude of irreducible representations that

$$\begin{aligned}\|\hat{a}\| &= \sup\{\|\hat{a}(\pi)\| : \pi \in \mathrm{Irr}(A{:}H)\}\\ &= \sup\{\|\pi(a)\| : \pi \in \mathrm{Irr}(A{:}H)\}\\ &= \|a\|,\end{aligned}$$

a generalized Gelfand transform is an isometric *-isomorphism. □

The condition (d) enables us to extend each x in $\mathrm{A}[\mathrm{Irr}(A{:}H)]$ to an operator field $\tilde{x}$ on $\mathrm{Rep}(A{:}H)$ as follows ;

$$\tilde{x}(\pi) := U^* \int_Z^{\oplus} x(\pi(\zeta))\, d\mu(\zeta)\, U$$

for every non-zero $\pi \in \mathrm{Rep}(A{:}H)$, where

$$U\,\pi\,U^* = \int_Z^{\oplus} \pi(\zeta)\, d\mu(\zeta)$$

is an irreducible decomposition of π over $\mathrm{Irr}(A{:}H)$. For the zero representation O, we define $\tilde{x}(O)$ to be the zero operator on H.

Now we are going to verify that for every $x \in \mathrm{A}[\mathrm{Irr}(A{:}H)]$, $\tilde{x}$ is an admissible operator field on $\mathrm{Rep}(A{:}H)$ which was defind and investigated in [4].

Lemma 1. *If* $V\pi V^* = \pi_1 \oplus \pi_2$ *for* π, π_1, $\pi_2 \in \mathrm{Rep}(A{:}H)$, *it holds*

$$V\,\tilde{x}(\pi)\,V^* = \tilde{x}(\pi_1) \oplus \tilde{x}(\pi_2)$$

for every $x \in \mathrm{A}[\mathrm{Irr}(A{:}H)]$.

Proof. Let

$$U_i\,\pi_i\,U_i^* = \int_{Z_i}^{\oplus} \pi_i(\zeta)\, d\mu_i$$

be an irreducible decomposition of π_i over $\mathrm{Irr}(A{:}H)$ $(i = 1, 2)$ then

$$\begin{aligned}&(U_1 \oplus U_2)V\,\pi\,V^*(U_1 \oplus U_2)^* \\ &= \int_{Z_1}^{\oplus} \pi_1(\zeta)\, d\mu_1(\zeta) \oplus \int_{Z_2}^{\oplus} \pi_2(\zeta)\, d\mu_2(\zeta)\end{aligned}$$

is an irreducible decomposition of π over $\mathrm{Irr}(A{:}H)$. Hence if $x \in \mathrm{A}[\mathrm{Irr}(A{:}H)]$ we have

$$\begin{aligned}&(U_1 \oplus U_2)V\,\tilde{x}(\pi)\,V^*(U_1 \oplus U_2)^* \\ &= \int_{Z_1}^{\oplus} x(\pi_1(\zeta))\, d\mu_1(\zeta) \oplus \int_{Z_2}^{\oplus} x(\pi_2(\zeta))\, d\mu_2(\zeta) \\ &= U_1\,\tilde{x}(\pi_1)\,U_1^* \oplus U_2\,\tilde{x}(\pi_2)\,U_2^* \\ &= (U_1 \oplus U_2)(\tilde{x}(\pi_1) \oplus \tilde{x}(\pi_2))(U_1 \oplus U_2)^*.\end{aligned}$$

Consequently, $V\,\tilde{x}(\pi)\,V^* = \tilde{x}(\pi_1) \oplus \tilde{x}(\pi_2)$. □

Corollary. *If* $\pi_1, \pi_2 \in \mathrm{Rep}(A{:}H)$ *and* $\pi_1 = V^*\,\pi_2\,V$, *then it holds*

$$\tilde{x}(\pi_1) = V^*\,\tilde{x}(\pi_2)\,V$$

for every $x \in \mathrm{A}[\mathrm{Irr}(A{:}H)]$.

Proof. Put $\pi = \pi_1$, $\pi_1 = \pi_2$, $\pi_2 = O$ in the preceding Lemma. □

Lemma 2. *For every* $x \in \mathrm{A}[\mathrm{Irr}(A{:}H)]$, *it holds*

$$p_\pi\,\tilde{x}(\pi)\,p_\pi = \tilde{x}(\pi)$$

for each $\pi \in \mathrm{Rep}(A{:}H)$.

Proof. Let

$$U\,\{\pi, H_\pi\}\,U^* = \int_Z^{\oplus} \{\pi(\zeta), H_\pi(\zeta)\}\, d\mu(\zeta)$$

be an irreducible decomposition of π over $\mathrm{Irr}(A{:}H)$ then we have

$$p_\pi = U^* \int_Z^{\oplus} p_\pi(\zeta)\, d\mu(\zeta)\, U$$

Hence by virture of the condition (a), for every $x \in \mathrm{A}[\mathrm{Irr}(A{:}H)]$ we have

$$\begin{aligned}&p_\pi\,\tilde{x}(\pi)\,p_\pi \\ &= U^* \int_Z^{\oplus} p_\pi(\zeta)\, x(\pi(\zeta))\, p_\pi(\zeta)\, d\mu(\zeta)\, U \\ &= U^* \int_Z^{\oplus} x(\pi(\zeta))\, d\mu(\zeta)\, U \\ &= \tilde{x}(\pi). \quad \square\end{aligned}$$

Lemma 3. *For every* $x \in \mathrm{A}[\mathrm{Irr}(A{:}H)]$, *it holds*

$$\sup\{\|\tilde{x}(\pi)\| : \pi \in \mathrm{Rep}(A{:}H)\} = \|x\|.$$

Proof. Let $\pi \in \mathrm{Rep}(A{:}H)$ and

$$U\,\pi\,U^* = \int_Z^{\oplus} \pi(\zeta)\, d\mu(\zeta)$$

be an irreducible decomposition of π over $\mathrm{Irr}(A{:}H)$. Then we have

$$\begin{aligned}\|\tilde{x}(\pi)\| &= \|\int_Z^{\oplus} x(\pi(\zeta))\, d\mu(\zeta)\| \\ &\le \operatorname{ess\,sup}\{\|x(\pi(\zeta))\| : \zeta \in \mathrm{Z}\} \\ &\le \sup\{\|x(\sigma)\| : \sigma \in \mathrm{Irr}(A{:}H)\} \\ &= \|x\|.\end{aligned}$$

Now put $\|\tilde{x}\| := \sup\{\|\tilde{x}(\pi)\| : \pi \in \mathrm{Rep}(A{:}H)\}$, then we have $\|\tilde{x}\| \le \|x\|$. Since it is obvious $\|\tilde{x}\| \ge \|x\|$, it holds $\|\tilde{x}\| = \|x\|$. □

Recall that $\tilde{A}$ donotes the C^*-algebra of all admissible operator fields on $\mathrm{Rep}(A{:}H)$. In [4] it was shown that $\tilde{A}$ can be faithfully represented as the enveloping von Neumann algebra of A.

Proposition 2. *For every* $x \in \mathrm{A}[\mathrm{Irr}(A{:}H)]$, $\tilde{x}$ *is an admissible operator field on* $\mathrm{Rep}(A{:}H)$

and the map. A[Irr(A:H)] $\ni x \mapsto \tilde{x} \in \tilde{A}$ *is an isometric *-isomorphism.*

Proof. By the previous Lemmas 1,2 and Corollary it follows that $\tilde{x}$ is an admissible operator field on Rep(A:H) for every $x \in$ A[Irr(A:H)].

It follows immediately by direct computations that for each $\pi \in$ Rep(A:H)

$$\begin{aligned}(\widetilde{\alpha x+\beta y})(\pi) &= \alpha\,\tilde{x}(\pi)+\beta\,\tilde{y}(\pi),\\ (\widetilde{xy})(\pi) &= \tilde{x}(\pi)\,\tilde{y}(\pi) \quad \textit{and}\\ \widetilde{x^*}(\pi) &= (\tilde{x}(\pi))^*\end{aligned}$$

for every x, $y \in$ A[Irr(A:H)] and every complex numbers α, β. These assert that the map. $x \mapsto \tilde{x}$ is a *-homomorphism of A[Irr(A:H)] into $\tilde{A}$. Proposition now follows from Lemma 3. □

Hence every $x \in$ A[Irr(A:H)] can be considered as an element of A^{**}. The conjugate space A^* is represented as $\{\omega(\pi;\xi,\eta) : \pi \in \mathrm{Rep}(A{:}H), \xi,\eta \in H_\pi\}$, where $\langle \omega(\pi;\xi,\eta), a\rangle = (\pi(a)\xi|\eta)$ for every $a \in A$.

The explicit dual relation of $x \in$ A[Irr(A:H)] $\subseteq A^{**}$ and $\omega(\pi;\xi,\eta) \in A^*$ is as follows ;

$$\begin{aligned}\langle x, \omega(\pi;\xi,\eta)\rangle &= \int_Z (x(\pi(\zeta))\xi(\zeta)|\eta(\zeta))\,d\mu(\zeta)\\ &= (\tilde{x}(\pi)\xi|\eta)\\ &= \langle \tilde{x}, \omega(\pi;\xi,\eta)\rangle,\end{aligned}$$

where

$$U\{\pi, H_\pi\}U^* = \int_Z^\oplus \{\pi(\zeta), H_\pi(\zeta)\}\,d\mu(\zeta)$$

is an irreducible decomposition of π over Irr(A:H).

3. A strong duality theorem

As in [4], the space Rep(A:H) is equipped with the topology of pointwise strong convergence. It means that $\pi_\alpha \to \pi$ in Rep(A:H) if and only if for each $a \in A$, $\|\pi_\alpha(a)\xi - \pi(a)\xi\| \to 0$ for every $\xi \in H$.

It is equivalent to the topology of pointwise weak, σ -weak, σ -strong, σ -strong* convergence (c.f.[2]). Because of the separability of A and H, topological space Rep(A:H) is a Polish space, that is, it is complete, metrizable and separable. Hence sequences are enough for us to consider with respect to the topology of Rep(A:H) instead of nets. The space Irr(A:H) has the relative topology, it is our dual object of A.

Henceforth we consider the continuity of $x \in$ A[Irr(A:H)] on Irr(A:H) which is compatible with the direct integrals of representations.

Definition 2. Let C[Irr(A:H)] denote the set of all $x \in$ A[Irr(A:H)] satisfies the following condition (e);

(e) If for each $a \in A$ a sequence

$$\left\{U_n^* \int_{Z_n}^\oplus \pi_n(\zeta)(a)\,d\mu_n(\zeta)\,U_n\right\}_{n\in N}$$

converges strongly to

$$U^* \int_Z^\oplus \pi(\zeta)(a)\,d\mu(\zeta)\,U$$

in $\mathfrak{B}(H)$, where $\pi_n(\zeta)$, $\pi(\zeta) \in$ Irr(A:H) a.e. then it holds that the sequence

$$\left\{U_n^* \int_{Z_n}^\oplus x(\pi_n(\zeta))\,d\mu_n(\zeta)\,U_n\right\}_{n\in N}$$

converges strongly to

$$U^* \int_Z^\oplus x(\pi(\zeta))\,d\mu(\zeta)\,U$$

in $\mathfrak{B}(H)$.

Since each $\pi \in$ Irr(A:H) is itself an irreducible decomposotion over Irr(A:H), we have

Cororally. *For every* $x \in$ C[Irr(A:H)], x *is continuous on* Irr(A:H).

Lemma 4. *For every* $x \in$ C[Irr(A:H)], $\tilde{x}$ *is continuous on* Rep(A:H).

Proof. Let $\pi_n \to \pi$ in Rep(A:H) and let

$$U_n^* \int_{Z_n}^\oplus \pi_n(\zeta)\,d\mu_n(\zeta)\,U_n, \quad U^* \int_Z^\oplus \pi(\zeta)\,d\mu(\zeta)\,U$$

be irreducible decompositions of π_n, π over Irr(A:H). Then for each $a \in A$ the sequence

$$\left\{U_n^* \int_{Z_n}^\oplus \pi_n(\zeta)(a)\,d\mu_n(\zeta)\,U_n\right\}_{n\in N}$$

converges strongly to

$$U^* \int_Z^\oplus \pi(\zeta)(a)\,d\mu(\zeta)\,U$$

in $\mathfrak{B}(H)$. By the condition (e), it follows for every $x \in$ C[Irr(A:H)] that the sequence

$$\{\tilde{x}(\pi_n)\} = \left\{U_n^* \int_{Z_n}^\oplus x(\pi_n(\zeta))\,d\mu_n(\zeta)\,U_n\right\}$$

converges strongly to

$$\tilde{x}(\pi) = U^* \int_Z^{\oplus} x(\pi(\zeta))\, d\mu(\zeta)\, U.$$

The statement now follows. □

Proposition 3. C[Irr(A:H)] *is a* C^*-*subalgebra of* A[Irr(A:H)].

Proof. Let $x, y \in$ C[Irr(A:H)], $\xi \in H$ and let keep the notations in the proof of Lemma 4. Since it holds that

$$\begin{aligned}&\|(\widetilde{xy}(\pi_n) - \widetilde{xy}(\pi))\xi\| \\ &\leq \|y\|\|(\tilde{x}(\pi_n) - \tilde{x}(\pi))\xi\| + \|x\|\|(\tilde{y}(\pi_n) - \tilde{y}(\pi))\xi\|,\end{aligned}$$

It follows that xy belongs to C[Irr(A:H)]. The rests of algebraic parts are obvious.

Let C[Irr(A:H)]$\ni x_\alpha \to x \in$ A[Irr(A:H)]. Since it holds that

$$\begin{aligned}&\| \int_Z^{\oplus} \{x_\alpha(\pi(\zeta)) - x(\pi(\zeta))\}\, d\mu(\zeta) \| \\ &\leq \|x_\alpha - x\| \to 0,\end{aligned}$$

x belongs to C[Irr(A:H)]. So C[Irr(A:H)] is closed in A[Irr(A:H)]. □

Now we are in the position to state and to prove our strong duality theorem.

Theorem. *C^*-algebra A is isomorphic to the C^*-algebra* C[Irr(A:H)] *under a generalised Gelfand transform.*

Proof. In Proposition 1 it was proved that a generalised Gelfand transform; $A \ni a \mapsto \hat{a} \in$ A[Irr(A:H)], where $\hat{a}(\pi) = \pi(a)$ for every $\pi \in$ Irr(A:H), is an isometric *-isomorphism. It is obvious that $\hat{a}$ satisfies the conditions (e), that is, $\hat{a}$ belongs to C[Irr(A:H)] for every $a \in A$.

Contrary let $x \in$ C[Irr(A:H)] then it follows from Proposition 2 and Lemma 4 that $\tilde{x}$ is continuous admissible operator field on Rep(A:H). Takesaki's duality theorem assure us that there exists uniquely $a \in A$ such that $\tilde{x}(\pi) = \pi(a)$ for every $\pi \in$Rep(A:H). Hence it follows that $x(\pi) = \tilde{x}(\pi) = \pi(a) = \hat{a}(\pi)$ for every $\pi \in$Irr(A:H). This implies that $x = \hat{a}$.

Therefore a generalised Gelfand transform is a *-isomorphism of A onto the C^*-algebra C[Irr(A:H)]. □

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