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Positive Time-Frequency Distributions: A Least Square Approach and A Copula-based Approach

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Abstract

Although the Wigner-Ville distribution (WVD) has many desirable properties, it has drawbacks: First, it is not positive. Second, it has large amount of cross terms. We present a least squares method (LSM) for generating a positive time-frequency distributions (TFDs) which belong to the Cohen-Zaparovanny positive distribution class. The positive distribution is as close as possible in the mean square sense to an arbitrary TFD or to a selected template constructed from bilinear TFDs. A simple example of two tones is given to illustrate that it is possible to obtain a positive distribution very close to a WVD and is contaminated with small cross terms. We also compare another method for constructing positive TFDs which is based on copula theory. We finally show a copula-based TFD applying to EEG signal.

1 Introduction

There have been many investigations of TFDs, for Example, the WVD (1, 2), spectrogram(3, 4) and others (5, 6, 7, 8, 9, 10). Cohen introduced a generalized class of TFDs from which most of the TFDs previously proposed can be generated(11). The class is called Cohen's class.

Bilinear distributions that are members of Cohen's class have been used with considerable success as tools for analyzing the time-frequency structure of nonstationary signals, because the distributions have explicit correlation information about the signal. However, the bilinear distributions which belong to Cone's class can not be positive(12, 13, 14). Positivity is one of the most important properties of a distribution because it can be interpreted as a true probability distribution or a true energy distribution. Although the spectrogram is not strictly a member of Cohen's class because it does not yield the correct marginal densities, it is positive as well as bilinear in the signal and belongs to the expanded version of Cohen's class that includes distributions which do not necessarily satisfy the correct marginals. Further, the spectrogram indicates well where the signal energy is localized and it smooths out cross terms that appear in many bilinear distributions of multicomponent signals. It, however, has the crucial drawback of rough resolution in time or/and frequency.

Cohen and Zaparovanny(15) and Cohen and Posch (16) showed that there is a infinite set of positive distributions in Cohen's class that satisfy the correct marginal distributions, and they gave a constructive method of generating them. However, there has been no way to interpret the spectral features of the arbitrarily constructed positive distributions because they lack explicit correlation information about the signal. There are a few systematic procedures for choosing a desirable distribution from the class. The authors proposed a least squares method (LSM) in the preliminary report in Japanese(17). A minimum cross entropy method (MCE)

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was proposed by Loughlin et al.[18] . In their method, however, non-positive distributions like WVD cannot be used as a template while they are available in the LSM. They also proposed the least squares method with moments constraints[19] . Davy and Doucet introduced a non-iterative method, i.e. copula-based technique, for computing positive TFDs recently[20] . They establish connections between Cohen-Zaparovanny, Cohen-Posch theory of positive TFDs and copula theory in statistics. Both theories address same problem, namely how to construct a joint probability distribution with imposed marginals? Davy and Doucet showed that both of them are formally equivalent.

In this report, we present a procedure for generating a positive distribution in the Cohen-Zaparovanny's class that is as close as possible in the mean square sense to an arbitrary TFD or to a selected template constructed from bilinear TFDs. An simple example of two tones is presented and compared the positive TFD we present with a positive distribution generated by Cohen and Posch[16] , the WVD, the spectrogram and copula-based TFD. We also show an example of time-frequency analysis of an EEG signal using copula-based positive TFD.

2 Cohen-Zaparovanny's Class

Let $x(t)$ be a deterministic complex signal with finite energy and $X(f)$ be the Fourier transform of $x(t)$. The total energy E of $x(t)$ is

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df < \infty \tag{1}$$

Assume that the TFD $S(t, f)$ satisfies the correct marginal distributions

$$|x(t)|^2 = \int_{-\infty}^{\infty} S(t, f) df \tag{2}$$

$$|X(f)|^2 = \int_{-\infty}^{\infty} S(t, f) dt \tag{3}$$

Cohen gave a general expression of the distribution $S(t, f)$ as

$$S(t, f) = \int\int\int_{-\infty}^{\infty} e^{-j2\pi(vt+f\tau-vu)} f(v, \tau) x(u + \tau/2) x^*(u - \tau/2) dv d\tau du, \tag{4}$$

where * denotes conjugate complex and $f(v, \tau)$ is a kernel function that satisfies the following conditions

$$f(v, 0) = f(0, t) = 1. \tag{5}$$

Then the marginal conditions Eqs. (2) and (3) are satisfied. The set of the distributions generated from Eq.(4) is called Cohen's class. The set of distributions generated by the same equation without the restrictions Eq. (5) is called Cohen's class in the wide sense. By choosing different kernels, different distributions can be obtained at will. When the kernel is independent of the signal, the time-frequency distribution derived from Eq.(4) becomes bilinear. There is no bilinear distribution that satisfies both correct marginal distributions and positivity[12, 13, 14] . On the other hand, if the kernel is a functional of the signal, positive distributions that satisfy the marginals can be generated. Cohen et al. have actually shown how to generate them [15, 16] . Their method is as follows. For the sake of simplicity, let the total energy of the signal be equal to 1 ($E = 1$). Consider a bivariate function $\rho(p, q)$ which is equal to or greater than -1

on the unit square ($0 \leq p, q \leq 1$) of the variables p, q such that

$$\int_0^1 \rho(p,q)dp = \int_0^1 \rho(p,q)dq = 0 \tag{6}$$

$$\rho(p,q) \geq -1. \tag{7}$$

Let p and q be functions of t and f respectively and be defined by

$$p(t) = \int_{-\infty}^{t} |x(t')|^2 dt' \tag{8}$$

$$q(f) = \int_{-\infty}^{f} |X(f')|^2 df'. \tag{9}$$

The distribution $G(t, f)$ defined by

$$G(t,f) = |x(t)|^2 |X(f)|^2 \{1 + \rho(p(t), q(f))\} \tag{10}$$

is positive and satisfies the marginal conditions of Eqs. (2), (3). On the other hand, by letting the kernel $f(v, \tau)$ be

$$f(v,\tau) = \frac{\int\int_{-\infty}^{\infty} |x(t)|^2 |X(f)|^2 \{1 + \rho(p(t), q(f))\} e^{j2\pi(vt+f\tau)} dtdf}{\int_{-\infty}^{\infty} x(u+\tau/2)x^*(u-\tau/2)du}, \tag{11}$$

we obtain the expression of Eq.(10). Therefore $G(t, f)$ defined by Eq.(10) belongs to Cohen's class. We call the general class of positive distributions given by Eq.(10) Cohen-Zaparovanny's class. It is shown that all possible positive distributions are generated by Eq.(10)[21, 22] . In a positive distribution, information about the signal is explicit only in the marginals. The marginals, however, do not have any correlation information. Therefore, the correlation information must be added to $G(t, f)$ by way of $\rho(p, q)$, which can be chosen at will.

Undesirable cross terms can occur in some positive distributions. Consider the example of $\rho(p, q)$ that was used in[16]

$$\rho(p,q) = (2p-1)(2q-1). \tag{12}$$

Let the signal be a sine wave, which initially has a frequency ω_1 in the interval $(0, t]$, then is shut off in the interval $(t_l, t_2]$ and turned on again in the interval $(t_2, t_3]$ with a frequency ω_2. It is desirable that the distribution has positive values in both the region of $\{0 < t \leq t_1, \omega = \omega_1\}$ and the region of $\{t_2 < t \leq t_3, \omega = \omega_2\}$ and diminishes in the rest of the time-frequency domain. However, as illustrated in Figure 1, the positive distribution has undesirable non-zero values (cross terms) in the region of $(0 < t \leq t_1, \omega = \omega_2)$ and the region of $(t_2 < t \leq t_3, \omega = \omega_1)$. In the corresponding regions $\{p(0) < p \leq p(t_1), q = q(\omega_2)\}$ and $\{p(t_2) < p \leq p(t_3), q = q(\omega_1)\}$ on the $p-q$ plane, the condition $\rho(p, q) = -1$ must hold so that the positive distribution has zero values there. However, the function defined by Eq.(12) does not satisfy the condition. This is why the positive distribution has undesirable nonzero values in the regions of $(0 < t \leq t_1, \omega = \omega_2)$ and $(t_2 < t \leq t_3, \omega = \omega_1)$. Figure 2, by Cohen[14] , shows the results of some other typical bilinear distributions for almost the same signal as that used in Figure 1. Figures 2(a), (b), and (c) show WVD [1, 2] , the Rihaczek distribution[9] , and the Page distribution[5] respectively. In these figures, only the positive parts are plotted, and only the real parts of the Rihaczek distribution are plotted in (c). In all three distributions, undesirable cross terms occur. The cross terms shown in Figure 1 are similar to those of the Rihaczek distribution.

Figure 3 shows the positive distribution for a signal with the reversed sequence of the sine waves with the angular frequencies ω_1 and ω_2. The positive distributions shown in Figure 1

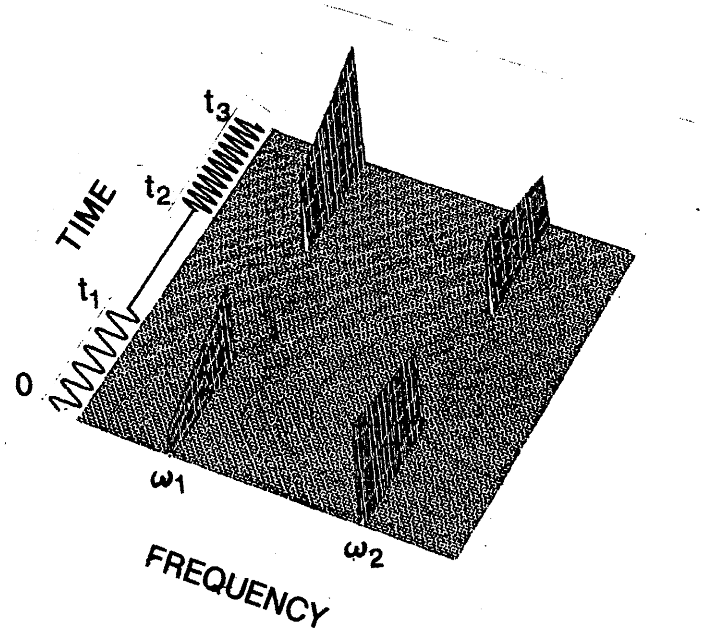

Figure 1: The positive distribution given by Eqs. (10) and (12) for the signal illustrated at left.

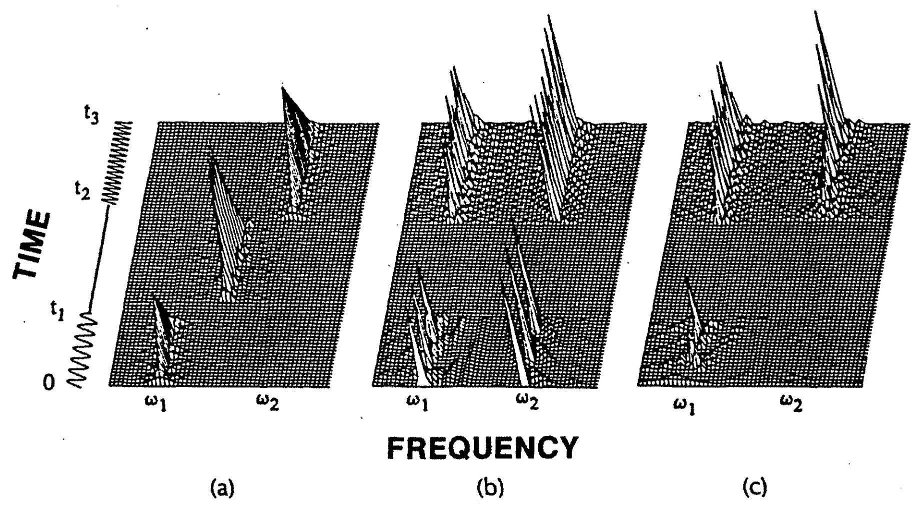

Figure 2: (a)Wigner, (b)Rihaczek, and (c)Page distributions for the signal illustrated at left [14]. Only the positive parts of the distributions are plotted[14].

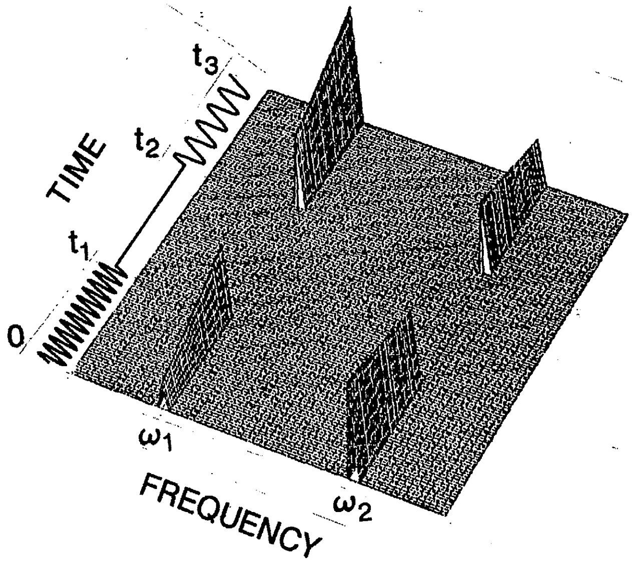

Figure 3: The positive distribution given by Eqs. (10) and (12) for the signal illustrated at left, which consists of the sequence of tones with angular frequencies ω_1 and ω_2 reverse to the signal in Figures 1 and 2.

and Figure 3 are much the same. The Rihaczek distribution displays the same undesirable property. If we let the kernel function $f(v, \tau)$ in Eq.(4) be itself the Wigner-Ville distribution of a time window function, the distribution obtained is positive everywhere[(23)] and we can obtain a distribution that does not have cross terms, because the cross terms are smoothed out. The distribution is called the spectrogram. The distribution, however, does not yield the correct marginal densities. It also has another crucial drawback; namely, it is subject to the time-frequency uncertainty.

Figure 4 shows the spectrogram of the same signal used in Figure 1. As illustrated in this figure, although the spectrogram has no cross terms, the frequency resolution is very poor because the duration of the time window is short.

3 The least squares method

As shown above, some positive distributions have undesirable cross terms as bilinear distributions has. The locations and shapes of the cross terms are directly related to $\rho(p, q)$. Further, the positive distributions acquire correlation information of the signal only by way of $\rho(p, q)$. Therefore, we have to be careful to choose $\rho(p, q)$ so that we obtain a positive distribution with no or small cross terms and with correct correlation information. However, few systematic procedures have been proposed for choosing a good $\rho(p, q)$. This section introduces a least squares method (LSM) for choosing a good $\rho(p, q)$ to generate a positive distribution in Cohen-Zaparovanny's class[(15)] , which is closest in the mean square sense to a given template. The

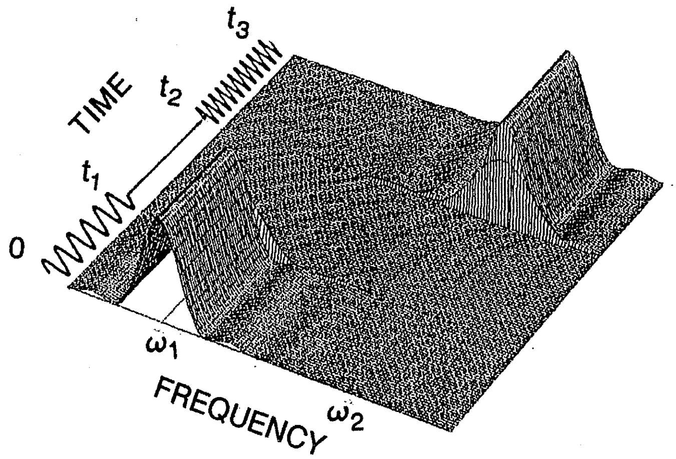

Figure 4: The spectrogram of the signal illustrated at left.

template should be chosen so that it has the correlation information of the signal in itself and other desirable properties like no or small cross terms and superior time-frequency resolution. The bilinear distributions inherently has correlation information of the signal, and some of them generally have desirable properties. Therefore, we choose a bilinear distribution $S(t, f)$ with desirable properties and use it as a template $T(t, f)$,

$$T(t, f) = S(t, f), \tag{13}$$

From the assumption $E = 1$, the next equation is satisfied.

$$\int\int_{-\infty}^{\infty} T(t, f)dtdf = 1. \tag{14}$$

We define the mean square error L as

$$L = \int\int_{-\infty}^{\infty} |G(t, f) - T(t, f)|^2 dtdf. \tag{15}$$

The positive distribution $G(t, f)$ is obtained by choosing $\rho(p, q)$ to minimize L.

The problem of choosing the optimum $\rho(p, q)$ can be solved by so-called non-linear programming, which needs iterative calculation. It is formulated as follows:

1. Put the objective function L as

$$L = \int\int_{-\infty}^{\infty} ||x(t)|^2|X(f)|^2\{1 + \rho(p(t), q(f))\} - T(t, f)|^2 dtdf. \tag{16}$$

2. Determine the initial function of $\rho(p, q)$.

3. Choose the function $\rho(p,q)$ so as to minimize the objective function L under the constraints

$$\int_0^1 \rho(p,q)dp = \int_0^1 \rho(p,q)dq = 0, \tag{17}$$

$$\rho(p,q) \geq -1, \tag{18}$$

4 Copula-based TFDs

In this section, we first show some basic definitions and properties related to copulas[24] . Then, we present how to find a time-frequency copula that captures time-frequency dependences of a signal[20] .

Definition 1 *A two-dimentional copula* $\mathbf{C}$ *is a function from* $[0,1]^2$ *to* $[0,1]$ *with following properties:*

1. $\mathbf{C}(u,0) = \mathbf{C}(0,v) = 0$ *for all* $(u,v) \in [0,1]^2$*;*
2. $\mathbf{C}(u,1) = u$ *and* $\mathbf{C}(1,v) = v$ *for all* $(u,v) \in [0,1]^2$*;*
3. *For all* $(u_1,u_2,v_1,v_2) \in [0,1]^4$ *such that* $u_1 \leq u_2$ *and* $v_1 \leq v_2$*,* $\mathbf{C}(u_2,v_2) - \mathbf{C}(u_2,v_1) - \mathbf{C}(u_1,v_2) + \mathbf{C}(u_1,v_1) \geq 0$.

Theorem 1 (Sklar's theorem [25]) *Let* $\mathbf{P}(x,y)$ *be a cumulative distribution with marginals* $\mathbf{X}(x)$ *and* $\mathbf{Y}(y)$*. Then there exists a copula* $\mathbf{C}$ *such that for all* $(x,y) \in \mathbf{R}^2$*,*

$$\mathbf{P}(x,y) = \mathbf{C}\left(\mathbf{X}(x), \mathbf{Y}(y)\right). \tag{19}$$

If $\mathbf{X}(x)$ *and* $\mathbf{Y}(y)$ *are continuous, then* $\mathbf{C}$ *is unique; otherwise,* $\mathbf{C}$ *is uniquely determined on* $\mathrm{Range}[\mathbf{X}(x)] \times \mathrm{Range}[\mathbf{Y}(y)]$*. Conversely, if* $\mathbf{X}(x)$ *and* $\mathbf{Y}(y)$ *are distributions and* C *is a copula, then* $\mathbf{P}(x,y)$ *defined by* (19) *is a joint distribution function with marginals* $\mathbf{X}(x)$ *and* $\mathbf{Y}(y)$*.*

The theorem justifies the use of copulas in the study of joint distributions with imposed marginals. It can be also inverted to express copulas in terms of a joint distribution and "quasi-inverses" of the two marginals.

Definition 2 *Let* $\mathbf{Q}$ *be a distribution function, A quasi-inverse* $\mathbf{Q}^{(-1)}$ *is such that*

$$\mathbf{Q}\left(\mathbf{Q}^{(-1)}(u)\right) = u, \quad \text{if } u \in \mathrm{Range}[\mathbf{Q}] \tag{20}$$

$$\mathbf{Q}^{(-1)}(u) = \inf\{x|\mathbf{Q}(x) \geq u\}, \quad \text{otherwise.} \tag{21}$$

Corollary 1 *Let* $\mathbf{P}(x,y)$ *be a cumulative distribution with marginals* $\mathbf{X}(x)$ *and* $\mathbf{Y}(y)$*, and let* $\mathbf{X}^{(-1)}(x)$ *and* $\mathbf{Y}^{(-1)}(y)$ *be quasi-inverses of* $\mathbf{X}(x)$ *and* $\mathbf{Y}(y)$*, and* $\mathbf{C}$ *be the corresponding copula. Then,*

$$\mathbf{C}(u,v) = \mathbf{P}\left(\mathbf{X}^{(-1)}(x), \mathbf{Y}^{(-1)}(y)\right). \tag{22}$$

This result means that one can construct a family of positive TFDs (or equivalently, joint distribution) whose marginals are fixed. It is, however, necessary to find a time-frequency copula that captures time-frequency dependences of a signal at hand in general. Davy and Doucet proposed a way of finding a time-frequency copula as below[20] .

Let $T(t,f)$ be a positive TFD which represents correctly the time-frequency contents of $x(t)$, but does not have correct marginals. Its cumulative distribution and marginals are denoted $\mathbf{T}(t,f)$, $\tilde{p}(t)$ and $\tilde{q}(f)$, respectively. Then, the time-frequency copula $\mathbf{C}^T$ of $\mathbf{T}(t,f)$ is such that (using corollary 1)

$$\mathbf{C}^T(u,v) = \mathbf{T}\left(\tilde{p}^{(-1)}(u), \tilde{q}^{(-1)}(v)\right). \tag{23}$$

A cumulative positive TFD $\mathbf{G}(t,f)$ of $x(t)$ which satisfies the correct marginals is then

$$\mathbf{G}(t,f) = \mathbf{C}^T\left(p(t), q(f)\right), \tag{24}$$

where $p(t)$ and $q(f)$ are the correct cumulative marginals defined by Eq.(8) and Eq.(9).

5 Examples

5.1 The least square approach

5.1.1 Two tones

A positive TFD using the least square technique is shown in Figure 5. The signal used here is the same one used in Figures 1 and 4. The WVD of the signal was used as a template $T(f,t)$. The initial distribution was $G(f,t) = |x(t)|^2|X(f)|^2$, that is, the initial function $\rho(p,q) = 0$ was used. In the non-linear program, we need to assign each point (t_i, f_j) in the equispaced digitized $t-f$ plane to a point (p_i, q_j) in the $p-q$ plane. Since p_i's and q_j's are not equispaced in most cases, we used the equispaced points closest to the original inequispaced points in order to save the computation time. The iterative calculation was repeated until the change in the magnitude of L was below a threshold.

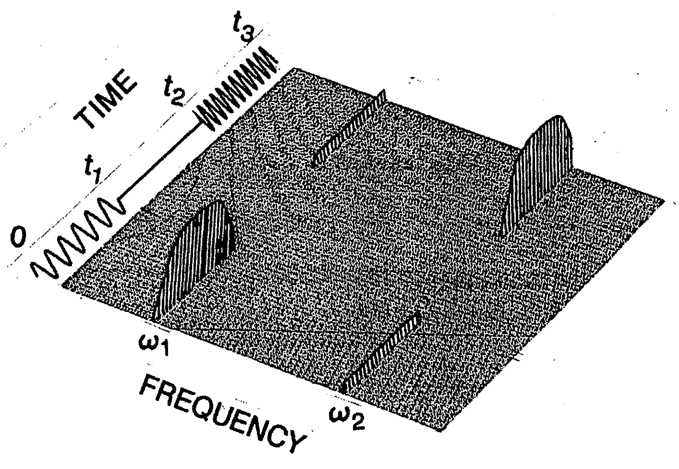

Figure 5: The positive distribution obtained by the LSM for the signal illustrated at left.

Although the resultant positive distribution shown in Figure 5 has cross terms in the same region as in Figure 1, they are very small and acceptable. The artificial cross terms are due to the quantization error in the step 3. In contrast, the cross terms of the template, i.e. WVD, shown in the region $\{t_1 < t \leq t_2, \omega = (\omega_1 + \omega_2)/2\}$ of Figure 2 disappear, because the values of the marginals $|x(t)|^2$ and/or $|X(f)|^2$ are zeros in that region.

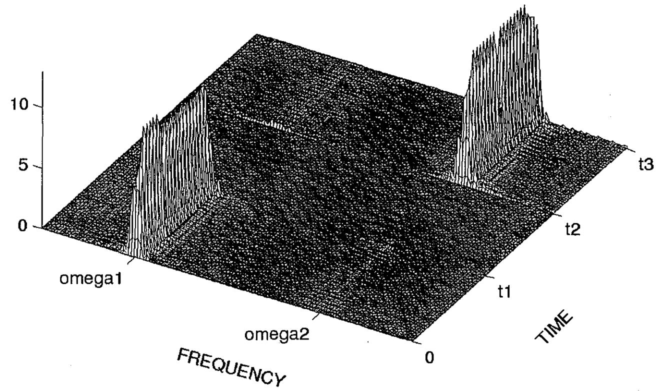

Figure 6: The positive distribution obtained by the copula-based method.

The time-frequency resolution of the resultant distribution is superior to that of the spectrogram in Figure 4. Further, the resolution was slightly better than that of the template. This is because the zero values of the marginals improved the resolution. Generally, the resolution of the template must be sustained in the resulting distribution unless the marginals $|x(t)|^2$ and/or $|X(f)|^2$ work well enough to sharpen the template. Therefore, it is important to choose a template with superior time-frequency resolution as well as small cross terms.

The resulting positive distribution shown in Figure 5 does not strictly satisfy the correct marginals. This is due to the quantization error, too. The shape of the resulting positive distribution is determined as closely as possible to the template under the constraints not of Eq.(17) and Eq.(18) but of its discrete version. Therefore, the shape of the resulting distribution and the resultant incorrect marginals depend on not only the shape of the template but the fineness of the quantization of the $t - f$ and $p - q$ planes. This explains why the shape of the distribution shown in Figure 5 slightly resembles the template WVD and has incorrect marginals.

5.2 Copula-based approach

5.2.1 Two tones

Figure 6 shows the copula-based TFD of the two tones signal which is used in previous section. We have used the cumulative spectrogram of the signal as a $\mathbf{T}(t, f)$ in Eq.(23). The cumulative TFD was then obtained by using Eq.(24).

In Figure 6, the vanishingly small cross terms are shown in the regions of $(0 < t \le t_1, \omega = \omega_2)$ and $(t_2 < t \le t_3, \omega = \omega_1)$. These are still more smaller than that of the positive TFD generated by the least square method.

The copula-based TFD has a very good time-frequency resolution. As shown in Figure 6, the copula-based TFD has comb-shaped energy distribution in the regions of $(0 < t \le t_1, \omega = \omega_1)$ and $(t_2 < t \le t_3, \omega = \omega_2)$. It is faithful reflection of the time $|x(t)|^2$ marginal, that is, the energy of sine wave becomes zero periodically.

5.2.2 EEG signal

We display a result of EEG analysis in this section. An EEG signal was measured from a normal healthy subject aged 22 for three minutes. During the measurement, the subject was ask to close his eyes and relax. Figure 7 shows the first 2.56 seconds of the EEG signal used in this report.

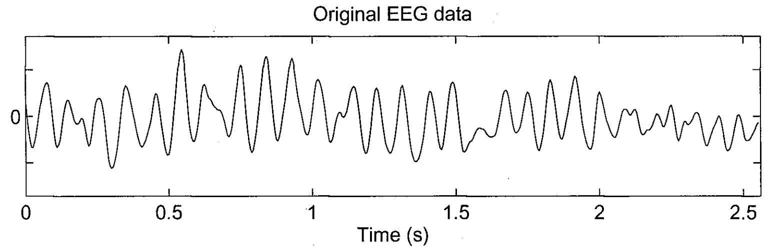

Figure 7: The EEG signal.

Figure 8(a) shows the spectrogram of the EEG signal which was calculated with 0.32 seconds sliding window and without overlap. Figure 8(b) represents the true frequency margin (solid line) and the cumulative frequency margin of the spectrogram (dotted line). Also, Figure 8(c) represents the true time margin (solid line) and the cumulative time margin of the spectrogram (dotted line). Note EEG signals are regarded as stochastic processes, hence we can not use $p(t)$ in Eq.(8) and $p(f)$ in Eq.(9) directly. In order to get the true marginals, we smoothed $p(t)$ and $q(f)$ with a simple moving average method. The sizes of the smoothing window are 75 (msec) in time and 1.18 (Hz) in frequency. The copula-based TFD of the EEG signal is shown in Figure 8(d). As shown in Figure 8, the spectrogram has poor time-frequency resolution while the copula-based TFD has excellent representation. The spectrogram is just enough to track the amount of the energy distribution of the α-wave band (8-12Hz) of the EEG signal. The copula-based TFD, however, can track the change of the peak frequency in the α-wave band.

6 Discussions

The undesirable properties, the cross terms and the incorrect marginals, of the positive TFD obtained by the least square approach are important issues to be settled. They can be reduced when an inequispaced sampling is adopted on the $p-q$ plane. Further, superior resolution of the sampling on both the $t-f$ plane and $p-q$ plane is recommended.

The template used in this example consists of only one bilinear TFD, that is $T(t,f) = S(t,f)$. It is, however, difficult to find a bilinear distribution that has both excellent cross term properties and superior time-frequency resolution at the same time. Therefore, the constructive template shown by the next equation must work more powerfully.

$$T(t,f) = cS_1(t,f)S_2(t,f)...S_n(t,f), \tag{25}$$

where c is the normalization constant and the component bilinear distributions are chosen so that they compensate for each other's defects. For example, a template produced by multiplying the WVD and the spectrogram will probably be recommended because WVD's superior time-frequency resolution compensates inferior time-frequency resolution of the spectrogram and excellent cross term property of the spectrogram cancels the large cross terms of the WVD. Some

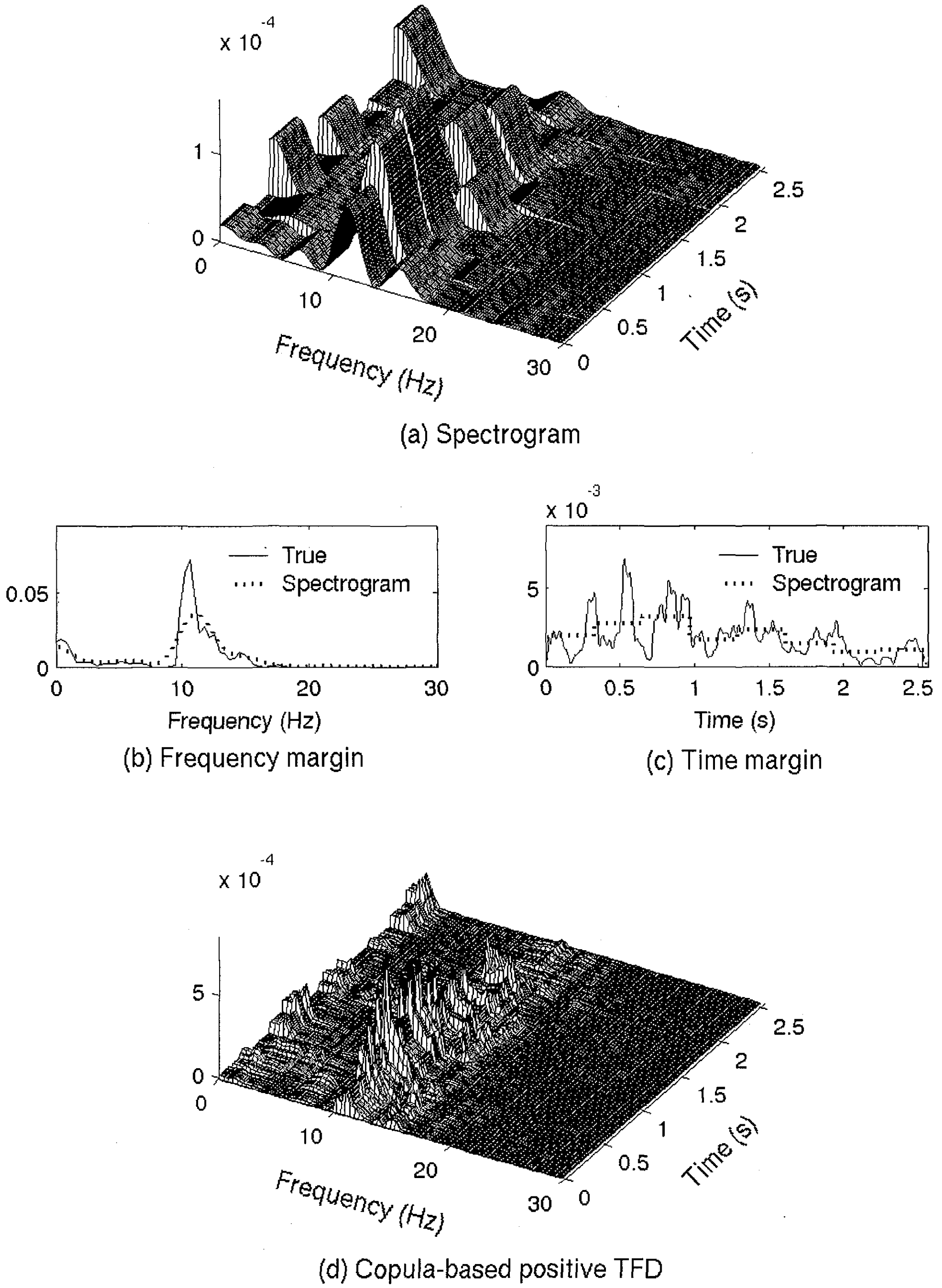

Figure 8: (a) Spectrogram, (b) Frequency margin, (c) Time margin, (d) Copula-based TFD of the EEG signal.

other distribution that have superior time-frequency resolution and excellent cross term properties, for example the Choi-Williams distribution[26] , the TFD with Cone-shaped kernels[27] , the TFD with compound kernel[28] are also recommended as constituent members.

A method of generating positive distributions by using the principle of minimum cross entropy (MCE) was proposed[18] Their idea is very similar to ours except that they minimize cross entropy instead of the mean squared distance. Although their idea is excellent, a distribution like the WVD that does not satisfy positivity cannot be used as a template. For a Gaussian-tapered complex 'chirp + tone' signal, they used the spectrogram as a prior estimate (template). To deal with the time/frequency resolution trade-off inherent in the spectrogram, they made the template by multiplying a wideband spectrogram and a narrowband spectrogram. Unfortunately, however, this method does not always work well. For example, for the chirp signal, narrowband spectrogram cannot be obtained no matter how long the time window is. Actually, Figure 1 in their paper shows that the frequency resolution of the chirp signal was not improved as well as the resolution of the tone signal was.

To overcome this problem, it is necessary to use other distributions with fine resolution. Unfortunately, such distributions cannot be positive and non-positive distributions cannot be used in the MCE as a template. This is a crucial defect of the MCE. In our LSM, however, both, positive distributions and non-positive distributions can be used in forming a template.

The cross terms also appear in the copula-based TFD, but it is negligible. It seems that cross terms are depend on the template used in the copula-based approach. For instance, the length of the windowing function of spectrograms or the type of positive joint distribution which is used as template would be important for the property. Moreover, consistency of the marginal condition seems to be related to the "quasi-inverse" function by Definition 2 in the case of the marginals are not continuous. Consequently, to get a distribution which satisfies the correct marginals more precisely, we propose Definition 3 instead of Definition 2

Definition 3 *Let* $\mathbf{Q}$ *be a distribution function, A quasi-inverse* $\mathbf{Q}^{(-1)}$ *is such that*

$$\mathbf{Q}\left(\mathbf{Q}^{(-1)}(u)\right) = u, \quad \text{if } u \in \text{Range}[\mathbf{Q}] \tag{26}$$

$$\mathbf{Q}^{(-1)}(u) = \arg\min_{x}\{|\mathbf{Q}(\hat{x}) - u|, |\mathbf{Q}(\check{x}) - u|\}, \quad \text{otherwise.} \tag{27}$$

where, $\hat{x} = \inf\{x|\mathbf{Q}(x) \geq u\}$ *and* $\check{x} = \sup\{x|\mathbf{Q}(x) \leq u\}$.

In addition, interpolation of the function $\tilde{p}(t)$ and $\tilde{q}(f)$ would improve the consistency of the marginal condition. The copula-based technique is noniterative method. This means that the computational cost of the copula-based technique is lower than that of iterative methods. The study of time-frequency copula is, however, just getting started. We need further investigation on these issues.

7 Conclusion

We proposed the LSM to choose the positive distribution in Cohen-Zaparovanny's class closest to the selected template distribution and showed an example to illustrate that we can obtain a positive distribution with intuitively desirable properties. In the example, the WVD was used as a template. We suggested that templates composed of multiple bilinear distributions make the LSM more powerful to deal with cross terms and time-frequency resolution problems. Most of the problems in our method are due to quantization errors and must be solved by making the sampling on both the $t-f$ plane and $p-q$ plane finer. The problem of the convergence in our iterative procedure is still open. The technique based on copula theory[20] was presented and we showed the examples, i.e. copula-based TFDs of a two tones signal and an EEG signal, too. Furthermore, we suggested that the modified definition of "quasi-inverse" function and interpolation of the marginals improve the consistency of the marginal condition.

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和文抄録

正値時間一周波数分布: 最小2乗法ならびにCopulaによる構成法

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Wigner-Ville 分布は信号の時間一周波数エネルギー分布としての好ましい特性を数多く持っているが、その一方で欠点もある。一つは、正値性を満さないことであり、もう一つは、クロス項が大きいことである。そこで、我々は Cohen-Zaparovanny のクラスに属する正値時間一周波数分布を最小2乗法によって構成する方法を提案する。この方法で得られる正値時間一周波数分布は、双線形型の正値時間一周波数分布から得られる分布、あるいは、任意のテンプレート分布に平均2乗誤差の意味で最も近い分布である。Wigner-Ville 分布をテンプレートとして、上述の正値時間一周波数分布が得られること、また、得られた分布には、僅かではあるがクロス項が存在することを簡単なシミュレーション例を用いて示した。また、Copula 理論に基く正値時間一周波数分布の構成法も取り上げ、これと比較した。最後に、実信号である脳波の正値時間一周波数分布についても示した。