PAPER

From vector analysis to fluid dynamics

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概要：音楽と揺らぎの研究[17]中に、流体力学の基礎的な部分の展開が些か分明でないことに注目し、本論文では、そのいくつかをより明確化する．それには、連鎖律、微分形式、一般のストウクスの定理をうまく組み合わせることで達成される．回転と湧き出しの概念を定理3.1で定式化する．さらに、２次元の流れの場合には、複素関数論を用いて同定理が導出できることを示す

Abstract：

キーワード：流体力学、発散、回転、一般のストウクスの定理、グリーンの公式、ガウスの発散定理、微分形式、連鎖律
Key words：fluid mechanics, divergence, rotation, general Stokes' theorem, Green's theorem, Gauss divergence theorem, differential forms, chain rule

1. Vector-valued functions

1.1 Differentiability

Let a vector-valued function $\boldsymbol{y}=\boldsymbol{y}(\boldsymbol{x})$ in a vector argument $\boldsymbol{x}=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}\in D\subset\mathbb{R}^n$ be given by

(1.1) $$\begin{pmatrix}y_1\\y_2\\\vdots\\y_m\end{pmatrix}=\boldsymbol{y}=\boldsymbol{y}(\boldsymbol{x})=\begin{pmatrix}y_1(\boldsymbol{x})\\y_2(\boldsymbol{x})\\\vdots\\y_m(\boldsymbol{x})\end{pmatrix}=\begin{pmatrix}y_1(x_1,x_2,\cdots,x_n)\\y_2(x_1,x_2,\cdots,x_n)\\\cdots\\y_m(x_1,x_2,\cdots,x_n)\end{pmatrix},$$

which is equivalent to the system of equations

(1.2) $$\begin{cases}y_1=y_1(\boldsymbol{x})=y_1(x_1,x_2,\cdots,x_n)\\y_2=y_2(\boldsymbol{x})=y_2(x_1,x_2,\cdots,x_n)\\\cdots\\y_m=y_m(\boldsymbol{x})=y_m(x_1,x_2,\cdots,x_n).\end{cases}$$

Definition 1. *The vector-valued function $\boldsymbol{y}=\boldsymbol{y}(\boldsymbol{x})$ is said to be totally differentiable (or Fréchet differentiable) at $\boldsymbol{x}$ if*

(1.3) $$\boldsymbol{y}(\boldsymbol{x}+\boldsymbol{h})=\boldsymbol{y}(\boldsymbol{x})+A\boldsymbol{h}+o(|\boldsymbol{h}|)$$

as $\boldsymbol{h}\to\boldsymbol{o}$,, i.e. $|\boldsymbol{h}|\to 0$.. Here A is an (m,n)-matrix and is called the gradient *(or the Jacobi matrix) of $\boldsymbol{y}$, denoted $\nabla\boldsymbol{y}$..*

(1.4) $$\nabla\boldsymbol{y}=\nabla\begin{pmatrix}y_1\\y_2\\\vdots\\y_m\end{pmatrix}=\begin{pmatrix}\nabla y_1\\\nabla y_2\\\vdots\\\nabla y_m\end{pmatrix}=\begin{pmatrix}\frac{\partial y_1}{\partial x_1}&\frac{\partial y_1}{\partial x_2}&\cdots&\frac{\partial y_1}{\partial x_n}\\\frac{\partial y_2}{\partial x_1}&\frac{\partial y_2}{\partial x_2}&\cdots&\frac{\partial y_2}{\partial x_n}\\\cdots\\\frac{\partial y_m}{\partial x_1}&\frac{\partial y_m}{\partial x_2}&\cdots&\frac{\partial y_m}{\partial x_n}\end{pmatrix}$$

In the case of a scalar function $\boldsymbol{y}=y=y(x_1,\cdots,x_n)$, the increment term $A\boldsymbol{h}$ in (1.1) amounts to

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(1.5) $$\nabla y\mathrm{d}\boldsymbol{x} = \frac{\partial y}{\partial x_1}\mathrm{d}x_1 + \cdots \frac{\partial y}{\partial x_n}\mathrm{d}x_n$$

on writing $\boldsymbol{h} = \mathrm{d}\boldsymbol{x} = (\mathrm{d}x_1, \cdots, \mathrm{d}x_n)$.

1.2　Chain rule

Theorem 1.1. Suppose (1.1) and

(1.6) $$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_\ell \end{pmatrix} = \boldsymbol{z} = \boldsymbol{z}(\boldsymbol{y}) = \begin{pmatrix} z_1(\boldsymbol{y}) \\ z_2(\boldsymbol{y}) \\ \vdots \\ z_\ell(\boldsymbol{y}) \end{pmatrix} = \begin{pmatrix} z_1(y_1, y_2, \cdots, y_m) \\ z_2(y_1, y_2, \cdots, y_m) \\ \cdots \\ z_\ell(y_1, y_2, \cdots, y_m) \end{pmatrix},$$

are differentiable. Then the composite function

(1.7) $$(\boldsymbol{z} \circ \boldsymbol{y})(\boldsymbol{x}) = \boldsymbol{z}(\boldsymbol{y}(\boldsymbol{x})) = \begin{pmatrix} z_1(y_1(x_1, \cdots, x_n), \cdots) \\ z_2(y_1(x_1, \cdots, x_n), \cdots) \\ \cdots \\ z_\ell(y_1(x_1, \cdots, x_n), \cdots) \end{pmatrix}$$

is differentiable and the gradient is given as the product of the gradients

(1.8) $$\nabla(\boldsymbol{z} \circ \boldsymbol{y}) = \nabla\boldsymbol{z}\nabla\boldsymbol{y},$$

i.e. the chain rule holds true which reads componentwisely,

(1.9) $$\frac{\partial z_i}{\partial x_j} = \sum_{k=1}^{m} \frac{\partial z_i}{\partial y_k}\frac{\partial y_k}{\partial x_j}.$$

2. Vector analysis

Definition 2. *For a vector-valued function* $\boldsymbol{f} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \in C^1(X)$ *(i.e.* $P, Q, R \in C^1(X)$*), we define its* **divergence** $(\mathrm{div}\boldsymbol{f})$ *and curl* $(\mathrm{curl}\boldsymbol{f})$ *(also called* **rotation** $\mathrm{rot}\boldsymbol{f}$*) by*

(2.1) $$\mathrm{div}\boldsymbol{f} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \boldsymbol{f}$$

and

(2.2) $$\mathrm{curl}\boldsymbol{f} = \begin{pmatrix} \begin{vmatrix} \frac{\partial}{\partial y} & Q \\ \frac{\partial}{\partial z} & R \end{vmatrix} \\ -\begin{vmatrix} \frac{\partial}{\partial x} & P \\ \frac{\partial}{\partial z} & R \end{vmatrix} \\ \begin{vmatrix} \frac{\partial}{\partial x} & P \\ \frac{\partial}{\partial y} & Q \end{vmatrix} \end{pmatrix} = \begin{pmatrix} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{pmatrix} = \nabla \times \boldsymbol{f}$$

Theorem 2.1. (General Stokes' theorem) *Let* $\mathcal{M}$ *be a* $k+1$*-dimensional manifold,* $\partial\mathcal{M}$ *its boundary and let* ω *be a differential form of degree* k *in* n *variables. Then the identity*

(2.3) $$\int \cdots \int_{\mathcal{M}} \mathrm{d}\omega = \int \cdots \int_{\partial\mathcal{M}} \omega$$

holds true or in the form

(2.4) $$\mathrm{d}\omega(\mathcal{M}) = \omega(\partial\mathcal{M}).$$

name	degree	number of var.
General Stokes' theorem	k	n
Green's theorem	1	2
Stokes' theorem	1	3
Gauss divergence theorem	2	3

Table 2.1. Special cases of general Stokes' theorem

In what follows we apply the following rule for computing the product of differential forms.

(2.5) $$\mathrm{d}y\mathrm{d}x = -\mathrm{d}x\mathrm{d}y, \quad \mathrm{d}x\mathrm{d}x = \mathrm{d}y\mathrm{d}y = 0.$$

The first equality in (2.5) may be thought of as representing the area with sign of an infinitesimal parallelogram and the second as the area of a degenerated parallelogram.

Theorem 2.2. *(Green's theorem) Let* $\mathcal{M} = \mathcal{D} \subset \mathbb{R}^2$ *be a domain,* $\partial\mathcal{D} \subset \mathbb{R}^2$ *its boundary and let* ω *be a* C^1 *class differential form of degree* 1 *in* 2 *variables:*

(2.6) $$\omega = P\mathrm{d}x + Q\mathrm{d}y.$$

Then we have

(2.7) $$\iint_{\mathcal{D}} \mathrm{d}\omega = \int_{\partial\mathcal{D}} \omega$$

or more concretely,

(2.8) $$\iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y = \int_{\partial\mathcal{D}} P\mathrm{d}x + Q\mathrm{d}y$$

Proof.

(2.9) $$\begin{aligned} \mathrm{d}\omega = \mathrm{d}P\mathrm{d}x + \mathrm{d}Q\mathrm{d}y &= \left(\frac{\partial P}{\partial x}\mathrm{d}x + \frac{\partial P}{\partial y}\mathrm{d}y\right)\mathrm{d}x \\ &+ \left(\frac{\partial Q}{\partial x}\mathrm{d}x + \frac{\partial Q}{\partial y}\mathrm{d}y\right)\mathrm{d}y \\ &= \frac{\partial P}{\partial y}\mathrm{d}y\mathrm{d}x + \frac{\partial Q}{\partial x}\mathrm{d}x\mathrm{d}y \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Theorem 2.3. *(Stokes' theorem) Let* $\mathcal{M} = \mathcal{S} \subset \mathbb{R}^3$ *be an orientable surface,* $\partial\mathcal{S} \subset \mathbb{R}^3$ *is a positively-oriented curve seen from the front side of* $\mathcal{S}$ *and let* ω *be a* C^1 *class differential form of degree* 1 *in* 3 *variables:*

(2.10) $$\omega = P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z.$$

Then we have

(2.11) $$\iint_{\mathcal{S}} \mathrm{d}\omega = \int_{\partial\mathcal{S}} \omega,$$

where the left side is the surface integral if $\mathcal{S}$ *is positively oriented and it is* -1 *times of the surface integral if* $\mathcal{S}$ *is negatively oriented. Or more concretely,*

(2.12)
$$\iint_S \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathrm{d}y\mathrm{d}z+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathrm{d}z\mathrm{d}x +\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y = \int_{\partial S} P\mathrm{d}x+Q\mathrm{d}y+R\mathrm{d}z.$$

Proof.

(2.13)
$$\begin{aligned}\mathrm{d}\omega &= \mathrm{d}P\mathrm{d}x+\mathrm{d}Q\mathrm{d}y+R\mathrm{d}z\\ &=\left(\frac{\partial P}{\partial x}\mathrm{d}x+\frac{\partial P}{\partial y}\mathrm{d}y+\frac{\partial P}{\partial z}\mathrm{d}z\right)\mathrm{d}x\\ &+\left(\frac{\partial Q}{\partial x}\mathrm{d}x+\frac{\partial Q}{\partial y}\mathrm{d}y+\frac{\partial Q}{\partial z}\mathrm{d}z\right)\mathrm{d}y\\ &+\left(\frac{\partial R}{\partial x}\mathrm{d}x+\frac{\partial R}{\partial y}\mathrm{d}y+\frac{\partial R}{\partial z}\mathrm{d}z\right)\mathrm{d}z\\ &=-\frac{\partial P}{\partial y}\mathrm{d}x\mathrm{d}y+\frac{\partial P}{\partial z}\mathrm{d}z\mathrm{d}x+\frac{\partial Q}{\partial x}\mathrm{d}x\mathrm{d}y-\frac{\partial Q}{\partial z}\mathrm{d}y\mathrm{d}z\\ &-\frac{\partial R}{\partial x}\mathrm{d}z\mathrm{d}x+\frac{\partial R}{\partial y}\mathrm{d}y\mathrm{d}z,\end{aligned}$$

which leads to (2.12).

Theorem 2.4. *(Gauss divergence theorem) Let* $\mathcal{M}=D\subset\mathbb{R}^3$ *be a domain,* $\partial D\subset\mathbb{R}^3$ *its oriented boundary surface with its front side positive and let* ω *be a* C^1 *class differential form of degree* 2 *in* 3 *variables:*

(2.14)
$$\omega = P\mathrm{d}y\mathrm{d}z+Q\mathrm{d}z\mathrm{d}x+R\mathrm{d}x\mathrm{d}y.$$

Then we have

(2.15)
$$\iiint_D \mathrm{d}\omega=\iint_{\partial D}\omega$$

or more concretely,

(2.16)
$$\iiint_D\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_{\partial D} P\mathrm{d}y\mathrm{d}z+Q\mathrm{d}z\mathrm{d}x+R\mathrm{d}x\mathrm{d}y$$

Proof.

(2.17)
$$\begin{aligned}\mathrm{d}\omega &= \mathrm{d}P\mathrm{d}y\mathrm{d}z+\mathrm{d}Q\mathrm{d}z\mathrm{d}x+dR\mathrm{d}x\mathrm{d}y\\ &=\left(\frac{\partial P}{\partial x}\mathrm{d}x+\frac{\partial P}{\partial y}\mathrm{d}y+\frac{\partial P}{\partial z}\mathrm{d}z\right)\mathrm{d}y\mathrm{d}z\\ &+\left(\frac{\partial Q}{\partial x}\mathrm{d}x+\frac{\partial Q}{\partial y}\mathrm{d}y+\frac{\partial Q}{\partial z}\mathrm{d}z\right)\mathrm{d}z\mathrm{d}x\\ &+\left(\frac{\partial R}{\partial x}\mathrm{d}x+\frac{\partial R}{\partial y}\mathrm{d}y+\frac{\partial R}{\partial z}\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y\\ &=\frac{\partial P}{\partial x}\mathrm{d}x\mathrm{d}y\mathrm{d}z+\frac{\partial Q}{\partial y}\mathrm{d}y\mathrm{d}z\mathrm{d}x+\frac{\partial R}{\partial z}\mathrm{d}z\mathrm{d}x\mathrm{d}y.\end{aligned}$$

The name comes from the fact that the left-hand side of (2.16) is equal to the integral of divergence defined by (2.1).

3. Rudiments of fluid mechanics

3.1 Divergence

Let $\boldsymbol{v}$ be the vector field of velocity of the fluid flowing in a domain $\Omega\subset\mathbb{R}^3$ and let $X\subset\Omega$ be a bounded closed domain with its boundary ∂X forming a surface. Let ρ denote the density (distribution) function of the fluid in Ω. Then $\boldsymbol{f}=\rho\boldsymbol{v}$ is the vector field describing the flow of mass distribution of the fluid.

At a point $X\ni P: \boldsymbol{x}=\boldsymbol{x}(u,v),$, the normal vector $\boldsymbol{n}$ is given by

(3.1)
$$\boldsymbol{n}=\frac{1}{|\boldsymbol{x}_u\times\boldsymbol{x}_v|}\boldsymbol{x}_u\times\boldsymbol{x}_v$$

and we write

(3.2)
$$f_n=\boldsymbol{f}\cdot\boldsymbol{n}$$

for the component of $\boldsymbol{f}$ in the direction of $\boldsymbol{n}$. Hence f_n *expresses the ratio of mass flowing out at P.*

Lemma 3.1. *Let* $\boldsymbol{f}=\begin{pmatrix}P\\Q\\R\end{pmatrix}\in C(\Omega)$ *be a continuous vector field in* $\Omega\subset\mathbb{R}^3$, $\omega=Pdydz+Qdzdx+Rdxdy$ *a differential form of degree* 2, $S\subset\Omega$ *a smooth surface given by* $S: \boldsymbol{x}=\boldsymbol{x}(u,v)$, $(u,v)\in\mathbb{R}^2$. *Then we have*

(3.3)
$$\iint_S (P\mathrm{d}y\mathrm{d}z+Q\mathrm{d}z\mathrm{d}x+R\mathrm{d}x\mathrm{d}y)=\iint_S f_n\,\mathrm{d}A,$$

where the right-hand side is the surface integral defined by

(3.4)
$$\iint_S f_n\,\mathrm{d}A=\iint_D f_n(\boldsymbol{x}(u,v))|\boldsymbol{x}_u\times\boldsymbol{x}_v|\,\mathrm{d}u\mathrm{d}v.$$

Proof. By (3.1) and (3.2), we have

(3.5)
$$f_n|\boldsymbol{x}_u\times\boldsymbol{x}_v|=\boldsymbol{f}\cdot(\boldsymbol{x}_u\times\boldsymbol{x}_v).$$

Substituting this, we have

(3.6)
$$\iint_S f_n\,\mathrm{d}A=\iint_D\left(P\frac{\partial(y,z)}{\partial(u,v)}-Q\frac{\partial(z,x)}{\partial(u,v)}+R\frac{\partial(x,y)}{\partial(u,v)}\right)\mathrm{d}u\mathrm{d}v,$$

which amounts to (3.3).

Therefore integral of f_n over all ∂X

(3.7)
$$Q=Q(\partial X):=\iint_{\partial X} f_n$$

expresses the **totality of masses flowing out of X in a unit time interval.**

Suppose there is no source or sink in X. Then since $Q(\partial X)$ indicates the rate of decrease of all masses in X, we have

(3.8)
$$\iint_{\partial X} f_n=-\frac{d}{dt}\iiint_X\rho.$$

By Lemma 3.1 and Theorem 2.4, the left-hand side is $\iiint_X \mathrm{div}(\boldsymbol{f})$::

(3.9)
$$\iint_{\partial X} f_n=\iiint_X \mathrm{div}(\boldsymbol{f}).$$

Hence

(3.10)
$$\iiint_X \mathrm{div}(\boldsymbol{f})=-\frac{d}{dt}\iiint_X\rho.$$

If $\frac{d\rho}{dt}$ is continuous, then we may change the integration and differentiation on the right of (3.8) and we obtain

(3.11) $$\iiint_X \operatorname{div}(\boldsymbol{f}) = -\iiint_X \frac{\partial\rho}{\partial t}.$$

Since (3.11) holds for any bounded domain X, we must have

(3.12) $$\operatorname{div}(\boldsymbol{f}) = -\frac{\partial\rho}{\partial t},$$

which is called the **equation of continuity** of fluid.

Definition 3. Fluid with constant density $\frac{\partial\rho}{\partial t} = 0$ is called **incompressible fluid**. For incompressible fluid, we have

(3.13) $$\operatorname{div}(\boldsymbol{f}) = 0.$$

3.2 Circulation

Let C be a smooth Jordan curve and let f_t be the component of $\boldsymbol{f}$ in the direction of the tangent vector $\boldsymbol{t} = \dot{x}(t)$ at a point $P \in C$::

(3.14) $$f_t = \boldsymbol{f}\cdot\frac{\boldsymbol{t}}{|\boldsymbol{t}|}.$$

Define the circulation of $\boldsymbol{f}$ around C by

(3.15) $$\Gamma = \Gamma(C) = \int_C f_t \, \mathrm{d}s.$$

Lemma 3.2. *Let* $\boldsymbol{f} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \in C(\Omega)$ *be a continuous vector field in* $\Omega \subset \mathbb{R}^3$, $\omega = Pdx + Qdy + Rdz$ *a differential form of degree* 1, $C \subset \Omega$ *a smooth curve given by* $C: \boldsymbol{x} = \boldsymbol{x}(s)$, $s \in [0, \Lambda]$.. *Then we have*

(3.16) $$\int_C \boldsymbol{f}\cdot \mathrm{d}\boldsymbol{x} = \int_C P\,\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = \int_C f_t \, \mathrm{d}s,$$

where the right-hand side is the 3*-dimensional line integral*

(3.17) $$\int_C \boldsymbol{f}\cdot \mathrm{d}\boldsymbol{x} = \int_C \omega$$

and the right-hand side is an integral in the arc length.

Proof. We find that the vector

$$\boldsymbol{v} = \boldsymbol{v}(s) = \boldsymbol{x}'(s)$$

satisfies $|\boldsymbol{v}| = 1$. Hence the component in (3.14) is $f_t = \boldsymbol{f}\cdot\boldsymbol{v}$. Hence

(3.18) $$\int_C f_t = \int_C \left(P\frac{dx}{ds} + Q\frac{dx}{ds} + R\frac{dx}{ds}\right)$$

which amounts to (3.17).

Combined with Stokes' theorem, Theorem 2.3, this gives

(3.19) $$\int_C f_t = \iint_D \mathrm{curl}(\boldsymbol{f})_{\boldsymbol{n}}.$$

From (3.9) and (3.19) we deduce the following

Theorem 3.1. *If If* $X \subset \mathbb{R}^3$ *is a bounded closed domain with its boundary* ∂X *forming a surface, then*

(3.20) $$Q = Q(\partial X) = \iint_{\partial X} f_n = \iiint_X \operatorname{div}(\boldsymbol{f}).$$

If $D \subset \mathbb{R}^3$ *is a domain with its boundary curve* $\partial D = C$, *then*

(3.21) $$\Gamma = \Gamma(C) = \int_C f_t = \iint_D \mathrm{curl}(\boldsymbol{f})_{\boldsymbol{n}}.$$

4. 2-dimensional flow

In this section we shall illustrate Theorem 2.1 by elucidating the results [7, pp.5-7] by complex analysis. Consider a 2-dimensional flow with its 3-velocity vector in the 3-dimensional space $\boldsymbol{v} = \begin{pmatrix} P \\ Q \\ 0 \end{pmatrix} \in C(D)$, where D is a domain with its boundary as a closed Jordan curve C.

(4.1) $$\operatorname{div}(\boldsymbol{v}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

and

(4.2) $$\mathrm{curl}(\boldsymbol{v}) = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix},$$

where $\omega = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ indicates the **vorticity.**

We consider the **complex velocity**

$$w = P - iQ$$

and integrate the function

(4.3) $$\mathrm{W} : \mathrm{curl}(\boldsymbol{v}) + i\operatorname{div}(\boldsymbol{v}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} + i\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)$$

over D:

(4.4) $$\iint_D W\,\mathrm{d}A = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A + i\iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\mathrm{d}A = \int_C P\,\mathrm{d}x + Q\mathrm{d}y + i\int_C -Q\mathrm{d}x + P\mathrm{d}y$$

by Green's theorem. The last curvilinear integrals may be combined as $\int_C w\,\mathrm{d}z$, where $z = x + iy$. I.e.. I.e.

(4.5) $$\iint_D W\,\mathrm{d}A = \int_C w\,\mathrm{d}z.$$

We also have

(4.6) $$w\mathrm{d}z = (v_t + iv_n)\mathrm{d}s,$$

where $v_t = \boldsymbol{v}\cdot\boldsymbol{t}$ is the component in the direction of the tangent vector and v_n the component in the direction of the tangent vector as defined in Section 3.

From (4.3), (4.5) and (4.6) we conclude that

(4.7) $$\iint_D (\mathrm{curl}(\boldsymbol{v}) + i\operatorname{div}(\boldsymbol{v}))\,\mathrm{d}A = \int_C (v_t + iv_n)\mathrm{d}s\,\mathrm{d}z.$$

Comparing the real and imaginary parts, we conclude the 2-dimensional version of Theorem 3.1.

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Record of the conducted interdsciplinary seminars

The 25th by Professor S. Kanemitsu:
On point groups--MS for "Computational Chemistry"
May 14th, 2014, Wed. 18:20-19:20, expected but not conducted. Only the MS was circulated

The 26th by Professor S. Kanemitsu:
Music, math. and fluctuations I
Jun. 25th, 2014, Wed. 18:20-19:20, Rm. 1303

The 27th by Professor S. Kanemitsu:
Music, math. and fluctuations II
Jul. 23rd, 2014, Wed. 18:20-19:20, Rm. 1303

The 28th by Professor T. Kaida:
On generalized constant-weight codes over GF(q) from a cyclic difference set and their properties
Sep. 24th, 2014, Wed. 18:20-19:20, Rm. 1305

The 29th by Professor S. Kanemitsu:
Besprechung von Professor Kaida's talk and Descartes'dream I
Oct. 22nd, 2014, Wed. 18:20-19:20, Rm. 1305

The 30th by Professor T. Matsuzaki
Rotational characteristics of feudal-lords spins
Jan. 28th, 2015, Wed. 18:20-19:20, Rm. 1305

The 31st sem. by Professor Kanemitsu
Music and fluctuations II
Apr. 22nd, 2015, Wed. 18:20-19:20, Rm. 1305

The 32nd sem. by Professor Kanemitsu
Music as mathematics of senses
Jun. 10th, 2015, Wed. 18:20-19:20, Rm. 1305

The 33rd sem. by Professor Kanemitsu
Special functions in number theory and science
Jul. 22nd, 2015, Wed. 18:20-19:00, Rm. 1305

The 34th sem. by Professor Kanemitsu
Special functions in number theory and science
Sep. 19th 2015, Wed. 18:20-19:00, Rm. 1305