PAPER

# Fluctuations in science and music

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**概要**：音楽は18－19世紀にその最高の高みに達したとよくいわれる。その原動力の一つは、音楽の作曲の堅牢さにあるといえる。その点、数学の堅牢さと通ずるものがある。ゲーテの言葉を借りれば、幾何学は凍れる音楽である、となる。ここで幾何学は数学にまで拡大される。それは、中世ヨーロッパの大学の高等教育で教えられていたのは７科目であり、そのうち３科目は、文法、論理学、修辞法であった。この３=triからtrivial（自明）ということばが派生した。数論、幾何学、天文学、音楽の４科目はより高等なものとして扱われ、quadriviaとよばれていた。ゲーテにとって幾何学=数学であったと思われる。音楽と数学のこの密接な関係に鑑みて、両者をつなぐ新理論の展開の場である関数空間を模索した結果、有限パワー信号（関数）の空間ではないかと考えるに至った。本論文はこれまで本セミナーシリーズで展開してきたフーリエ変換理論の頂点であり、生き物と死に物の境界に位置する対象研究の魁である。

**キーワード**：有限エネルギー信号、有限パワー信号、自己相関関数、フーリエ変換、音階、ゆらぎ

**Key words**：finite energy signals, finite power signals, autocorrelation, Fourier transform, musical scales, fluctuations

## 1. Introduction

### 1.1. HiFi vs. WiFi

Before the time of reproduction by digital apparatuses, sonic data, esp. music was always recorded in tapes (or records before the tape-recorders) and the principle was to reproduce as near as possible the original sound—**high fidelity**, hifi. However, in the time of computers, the wifi is prevailing, wide fidelity not high fidelity. Already in the times of records, there was a bud of such a tendency represented by Dolby system, which cuts higher frequencies in order to reduce the noise made by the record player.

There is wide-spread misunderstanding of the digital data that they reproduce the whole data, i.e. after analogue to digital transform $A/D$, certain processing conducted, and the resulting digital data $D$ remaining the same as the original digital data $D$, and finally by the digital analogue transform $D/A$, the output $A$ is obtained, but the output is not $A$ but $A'$ and the whole process gives only

(1.1) $A/D \to D/A'$,

where $A'$ is an **avatar** of $A$.

What is missing here is the fact that the output is **perceived by human senses** (ears in the case of sound) first and then interpreted according to the stored data in the brains. Therefore, e.g. for **tapping**, only the obtained data is essential and the tones or musical flavor is not relevant. Or in the case of pictures, the brains can be easily deceived. By the very principle of evolution, the human brains tend to recognize human faces in all objects as the humans were the only their allies against many fierce carnivorous predators. Thus,

"what is perceived by sight can be easily deceived by mind."

But the case of music can be different and digital music cannot deceive trained musical ears and many experienced worshipers of music cannot tolerate the processed music.

Digital reproduction looks like finding an avatar—wifi rather than re-creating an as near as the original—hifi.

The reason for such a sloppy misunderstanding lies rather deep in our modern culture influenced too much by the 20th century sectionalism and the local global principle to the effect



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that decomposing an object into smaller and smaller objects until one gets some simple governing principles, processing them in terms of modern science, and finally compiling them into something, which one is feasible to believe to be the original whole. This can be seen typically in molecular biology—Phenotype determines genotype, which cannot be true. One must bear in mind that the obtained model is a means to describes the object from one direction or other and cannot be the original.

In [6, pp.30-33] the principle of CD is stated. The pitch of the sound can be divided into $2^{16}=65536$ parts because a CD can record $16$ convex-concave points as one information and transforms into $0$ and $1$ signal. The CD reads these $16$ information $44100$ times per second. The reason for this depends on the assumption that human ears can hear the sound whose frequencies are up to $20kHz=20000Hz$. Since the sound with frequency $20000Hz$ oscillates $20000$ times per second and so more than this times of sampling is needed. And for stereo recording, we need twice as many, whence the sampling frequency $44100$. Thus the sampling (Nyquist) rate $\frac{\pi}{\tau}$ is $1/44100$ and the input digital signal is restored by the sampling theorem (Theorem 1.1 below) [12], [13] etc.

The assumption that all the frequencies higher than $20000Hz$ may be neglected is rather controversial. For recording simple conversation may not need more, but for music as supreme art, this omission can be a serious problem because what is missing is often more meaningful as art, cf. the next subsection. We recall that as soon as the real sound is transformed into digital signals, it is not the real signal but an approximation.



## 1.2. Real world sampling theorem

The following theorem ([17, Theorem 5.2, pp.119-120]), stated in [13], explains why the sampling rate may be taken as the **samples per cycle of the highest frequency**. Re call that a signal is called almost band-limited if the support of its Fourier transform is contained in a finite interval, say $[-\Omega, \Omega]$, $\Omega$ being called the band-length.

**Theorem 1.1.** (Real world sampling theorem) *If the function* $f(t)$ *is almost band-limited with band-length* $2\Omega$ *and* **almost time-limited** *with time limit* $2X$:

(1.2) $\int_{-\infty}^{X}|f(t)|\mathrm{d}t<\varepsilon,\ \int_{X}^{\infty}|f(t)|\mathrm{d}t<\varepsilon,$

then $f(t)$ *can be recovered to any desired accuracy from its sampled values at uniformly-spaced intervals* $|\Delta t|<\frac{1}{10\Omega}$ *apart:*

(1.3) $f(t)=\sum_{n=-M}^{M}f(n\Delta t)\frac{\sin(2\pi 5\Omega(t-n\Delta t))}{2\pi 5\Omega(t-n\Delta t)}+R_{\varepsilon}(t)$

*where*

(1.4) $|R_{\varepsilon}(t)|<<\varepsilon,\ M\Delta t>X,\ \Delta t<\frac{1}{10\Omega}.$

**Remark 1.1.** *An example of a function which is both almost band-limited and almost time-limited is the Gaussian function or its several-variables version*

(1.5) $$f(t)=e^{-at^2},\ a>0.$$

*This is a density function of the normal (Gaussian) distribution as well as an almost unique example of a rapidly decreasing function. Phenomena whose distribution is in terms of this density function is often referred to as Gaussian.*

*Comparing the sample frequency* $44100$ *with (1.4), it seems that* $\Omega=4410$ *only and that much lower frequencies are cut. We are going to analyze this aspect further on.*

## 1.3. Praise of the non-present

Not only in music but flower arrangement, it is said that what is important is rather the blank space surrounding the arranged flowers. We recall a poem of R. Kinoshita

Peony flowers
So stable and in full bloom
The solidness of the position
The flowers occupy.

–“As is true with all artifacts, completed ones are less interesting; those which are unfinished and left at that are more appealing and give liveliness.”
from *An hermite's miscellany* by K. Urabe.

Here of course, “an hermite's” is a play on words of math. flavor. For the word “hermit” is naturally associated with the name Hermite and since in French, the ‘h’ is not pronounced, the indefinite article is to be ‘an’. K. Urabe is more well-known as Kenko Yoshida.

–Hans describes the crystal of snow as “This is too regular. A living thing cannot be so symmetric. Things with such perfection smell of death. The Creator, it seems, has designed all that exists in his world in a fashion slightly deviated from perfect symmetry” –*Der Zauberberg* by Thomas Mann.

Also “In Praise of Shadows” by J. Tanizaki explores the Japanese sense of beauty—the subtle interplay of **shade and light** in several important aspects of Japanese life. I.e. it puts stress on what is missing. It's a pity that music is not referred to here.

## 2. Finite power signals

### 2.1. Finite energy and finite power signals

We follow [10, pp.240-264]. For a signal $f(t)$ its **average power** is defined by

(2.1) $$f^2=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(t)|^2\mathrm{d}t,$$

which is often denoted by some other symbols like power$(f)$ [12, (7.2.7)]. The class of functions with the average value $0$, called the class of **finite energy signals**, has been extensively studied and which clearly contains the subclass of **finite energy functions** for which

(2.2) $$\int_{-\infty}^{\infty}|f(t)|^2\mathrm{d}t<\infty,$$

i.e. $L^2$-functions (square integrable functions).

We consider the class of functions with non-zero average power, i.e. $0<f^2<\infty$, calling it the **finite power signals** class. Such functions do not have Fourier transforms in general. An important feature is that this class contains all periodic functions. Indeed, let $f(t)$ be a periodic function given by

(2.3) $$f(t)=\sum_{n=-\infty}^{\infty}a_n e^{in\omega_0 t}$$

with $\sum_{n=-\infty}^{\infty}|a_n|^2<\infty$, so that $f\in\ell^2$. Then

(2.4) $$f^2=\frac{2\pi}{\omega_0}\int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}}|f(t)|^2\mathrm{d}t=\sum_{n=-\infty}^{\infty}|a_n|^2.$$

The class of finite power signals also contains non-periodic function which are usually specified by some averages. The most common one is the **auto-correlation** defined by

(2.5) $$R_f(t)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(\tau)f(t+\tau)\mathrm{d}\tau.$$

In what follows we shall speak about this average.

### 2.2. Preliminarie

The theory of Fourier transform has been our main driving force throughout in our seminars and its rudiments have been given in our previous papers. Although there are some overlaps of stated results, the presentation is different from one paper to another. I.e. [12] is of functional analytic, [4] of distribution-theoretic, and [13] is centered around the Paley-Wiener theorem. And in this article, we take the lines of convolutions. Needless to say, the most important feature is that the transform can be inverted, known as the inverse Fourier transform or the Fourier integral theorem [12, p.70], [4, p.12], [13, p.10]. This corresponds to the most fundamental principle in science, the fundamental theorem of calculus, i.e. differentiation and integration are inverse operations. Note that after applying any transformation, one cannot go back, i.e. it is one-sided. Only Fourier transform is invertible (which includes the wavelet transform). We restate [Vista I, (7.24)][12, p.70] in correct form

(2.6) $$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega t}\hat{f}(\omega)\mathrm{d}\omega.$$

Here the Fourier transform is defined by

(2.7) $$F(\omega)=\hat{f}(\omega)=\int_{-\infty}^{\infty}e^{-1t\omega}f(t)\mathrm{d}t.$$

In [13, p.10] we put the factor $\frac{1}{\sqrt{2\pi}}$ in (2.7) and so in (2.6), the factor changes into $\frac{1}{\sqrt{2\pi}}$, giving a more symmetric form. We denote the correspondence furnished by (2.6) and (2.7) by

(2.8) $$f(t)\leftrightarrow F(\omega)=\hat{f}(\omega).$$

Thus we have

(2.9) $$\{\text{time domain}\}\leftrightarrow\{\text{frequency domain}\}$$

and we may work with the functions in the frequency, which is easier in many respects. However, recall the criticism by G. Matsumoto [7] on the sloppy application of frequency analysis to the living. Then (2.6) and (2.7) may be combined into the symmetry property

(2.10) $$F(t)=f(t)\leftrightarrow 2\pi f(-\omega)\ \text{or}\ f(t)=2\pi f(-t),$$

which is stated [12, p.70]

Expressing the Fourier transform $\hat{f}(\omega)$ as

(2.11) $$F(\omega)=\hat{f}(\omega)=A(\omega)e^{i\varphi(\omega)},$$

we call $A(\omega)>0$ the **Fourier spectrum** and $E(\omega):=A^2(\omega)=F(\omega)F(\omega)$ the **energy spectrum** of the signal $f(t)$, where the bar indicates the complex conjugation. $\psi(\phi mega)$ is called the phase angle. It follows that

(2.12) $$f(t)\leftrightarrow F(-\omega)=\bar{\hat{f}}(-\omega).$$

Note that

(2.13) $$A^2(\omega)=|\hat{f}|^2=\hat{f}(\omega)\bar{\hat{f}}(\omega)=F(\omega)F(\omega).$$

It follows that $\hat{f}(-\omega)=\bar{\hat{f}}(\omega)$ is necessary and sufficient for $f(t)$ to be real. This sometimes makes argument simpler and will be used occasionally.

In the theory, equally important notion is the convolution, which has been introduced in [4, Definition 5] and [13, Definition 1], which exists for square integrable functions.

(2.14) $$(f_1*f_2)(t)=\int_{-\infty}^{\infty}f_1(\tau)f_2(t-\tau)\mathrm{d}\tau.$$

The following theorem was stated as Exercise 4.1 [4]

**Theorem 2.1.** (Time convolution theorem) *The Fourier transform $\hat{f}(\omega)$ of the convolution $f(t)$ of two functions $f_1(t)$ and $f_2(t)$ is the product of the Fourier transforms $\hat{f}_1(\omega)$ and $\hat{f}_2(\omega)$. I.e.*

(2.15) $$(f_1*f_2)(t)=\int_{-\infty}^{\infty}f_1(\tau)f_2(t-\tau)\mathrm{dr}\leftrightarrow f_1(\omega)\cdot f_2(\omega)$$

*Proof.* In

(2.16) $$\hat{f}(\omega)=\int_{-\infty}^{\infty}e^{-i\omega t}\left(\int_{-\infty}^{\infty}f_1(\tau)f_2(t-\tau)\mathrm{dr}\right)\mathrm{d}t,$$

we change the order of integration. Then the inner integral in $t$ is

(2.17) $$\hat{f}(\omega)=\int_{-\infty}^{\infty}e^{-i\omega t}f_2(t-\tau)\mathrm{d}t,$$

which by the change of variable is seen to be $e^{-i\omega t}\hat{f}_2(\omega)$ Hence

(2.16) amounts to

(2.18) $$\hat{f}(\omega)=\int_{-\infty}^{\infty} f_1(\tau)e^{-i\omega t}\hat{f}_2(\omega)\mathrm{d}\tau\mathrm{d}t,$$

leading to (2.15). □

**Theorem 2.2.** (Frequency convolution theorem) *The Fourier transform* $\hat{f}(\omega)$ *of the convolution* $f(t)$ *of two functions* $f_1(t)$ *and* $f_2(t)$ *is the product of the Fourier transforms* $\hat{f}_1(\omega)$ *and* $\hat{f}_2(\omega)$. *I.e.*

(2.19) $$(f_1\ f_2)(t)\leftrightarrow\frac{1}{2\pi}f_1(\omega)\cdot f_2(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f_1(\tau)f_2(t-\tau)\mathrm{d}\tau.$$

This follows from Theorem 2.1 together with (2.10). Direct proof can be given in the same lines as that of Theorem 2.1.

The **Parseval identity** was stated as [12, (7.3.7)], which under (2.6) reads

(2.20) $$\int_{-\infty}^{\infty}|f(t)|^2\mathrm{d}t=\frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2\mathrm{d}\omega.$$

We shall prove a slight generalization of (2.20).

**Theorem 2.3.** *If*

(2.21) $$f_1(t)\leftrightarrow F_1(\omega),\ f_2(t)\leftrightarrow F_2(\omega),$$

*then*

(2.22) $$\int_{-\infty}^{\infty}f_1(t)f_2(t)\mathrm{d}t=\frac{1}{2\pi}\int_{-\infty}^{\infty}F_1(-\omega)F_2(\omega)\mathrm{d}\omega.$$

*Proof.* By Theorem 2.2,

(2.23) $$f_1(t)f_2(t)\leftrightarrow\frac{1}{2\pi}F_1(\omega)*F_2(\omega),$$

which means

(2.24) $$\frac{1}{2\pi}\int_{-\infty}^{\infty}F_2(\omega-w)F_2(w)\mathrm{d}w=\int_{-\infty}^{\infty}e^{-i\omega t}f_1(t)f_2(t)\mathrm{d}t$$

Eq. (2.24) with $\omega=0$ amounts to (2.22). □

Theorem 2.3 with $f_2=f_1$, $F_2=F_1$ leads to (2.20) on account of (2.12). If $f_1$, $f_2$ are real, then (2.24) may be written as

(2.25) $$\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{F}_1(\omega)F_2(\omega)\mathrm{d}\omega=\int_{-\infty}^{\infty}e^{-i\omega t}f_1(t)f_2(t)\mathrm{d}t$$

by (2.12) again.

**Remark 2.1.** (2.20) *gives justification of the name energy spectrum* $A^2(\omega)$ *under* (2.11). *Indeed, if we assume* $f(t)$ *is a voltage of a source across a resistance of* 1 *Ohm, then the left-hand side of* (2.20) *is the total energy delivered by the source. And by* (2.20), *it is equal to the area under the curve* $y=\frac{1}{2\pi}A^2(\omega)$.

## 2.3. Finite energy signals

As a starter for the study of finite power functions, we dwell on finite energy functions.

Suppose that $f(t)$ is a signal possessing the Fourier transform. The Fourier inversion of the energy spectrum $A^2(\omega)$, cf. (2.11), called the **autocorrelation**, is denoted by $\rho_f(t)$:

(2.26) $$\rho_f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}A^2(\omega)e^{i\omega t}\mathrm{d}\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}A^2(\omega)\cos\omega t\,\mathrm{d}\omega.$$

The autocorrelation for the finite energy signals is defined by

(2.27) $$\rho_f(t)=\int_{-\infty}^{\infty}f(t+\tau)f(\tau)\mathrm{d}\tau$$

corresponding to (2.5).

**Theorem 2.4.** (2.26) *coincides with* (2.27).

*Proof.* Suppose for simplicity that $f(\tau)$ is a real signal.

By (2.12) and (2.13), we have

(2.28) $$\rho_f\leftrightarrow A^2(\omega)=f(\omega)f(\omega)=F(\omega)F(\omega)=F(\omega)F(-\omega)$$

the last equality being due to realness of $f$.

By Theorem 2.10, we have $f(-t)\leftrightarrow F=F(-\omega)$. Hence by the time convolution theorem, Theorem 2.1,

(2.29) $$\rho_f(t)=f(t)*f(-t)=\int_{-\infty}^{\infty}f(t-\tau)f(-\tau)d\tau.$$

This is nothing but (2.27). For a complex signal, we merely separate real and imaginary parts and apply (2.29), completing the proof. □

Plainly we have

(2.30) $$\rho_f(-t)=\bar{\rho}(t)$$

and

(2.31) $$|\rho_f(t)|\le\rho_f(0)=\int_{-\infty}^{\infty}|f(t)|^2\mathrm{d}t$$

by the Cauchy-Schwarz inequality and (2.2).

## 2.4. Finite power signals

The argument goes almost parallel to that of §2.3. The autocorrelation $R_f(t)$ of a finite power signal $f(t)$ is defined by (2.5). First note that (2.1) may be written as

(2.32) $$f^2=\lim_{T\to\infty}\frac{1}{2T}\int_{-T-A}^{T+A}|f(t)|^2\mathrm{d}t=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(t+A)|^2\mathrm{d}t$$

for any constant $A$.

In the same way as we prove (2.31) using (2.32), we may prove that the autocorrelation (2.5) exists and

(2.33) $$|R_f(t)|\le R_f(0)$$

and that $R(t)$ is even: $R(-t)=R(t)$.

As in §3, the power spectrum $S_f(\omega)$ of a finite power signal $f$ is to be defined as the Fourier transform of the autocorrelation in (2.5):

(2.34) $$S(\omega)\leftrightarrow R(t)$$

or

(2.35) $$S(\omega)=\int_{-\infty}^{\infty}R(t)\cos\omega t\,\mathrm{d}t,$$

**Theorem 2.5.** *We have*

(2.36) $$S(\omega)=\lim_{T\to\infty}\frac{1}{2T}\left|\int_{-T}^{T}f(t)e^{-j\omega t}\mathrm{d}t\right|^2.$$

*Proof.* The lines of proof are exactly the same as those of Theorem 2.4 save for the smoothing and Helly's theorem. First we show that (2.36) is the limit of

(2.37) $$S_T(\omega)=\frac{1}{2T}|F_T(\omega)|^2$$

which is the average power $\overline{f_T}=S_T(\omega)$ of

(2.38) $$F_T(\omega)=\int_{-T}^{T}f(t)e^{-j\omega t}\mathrm{d}t.$$

Here $F_T(\omega)$ is the Fourier transform of $f_T$ which is the smoothing referred to in [4, Example 11,(ii)]:

(2.39) $$f_T(t)=f(t)r_T(t)=\begin{cases} f(t) & |t|<T \\ 0 & |t|\geq T \end{cases},$$

which is a time-limited version of $f$ and where $r_T(t)$ is the rectangular pulse function.

Since the right-hand side of (2.37) is equal to $\frac{1}{2T}F_T(\omega)$ $F_T(-\omega)$ correspondingly to (2.28), it follows by Theorem 1 that the inverse Fourier transform $R_T(t)$ of $S_T(\omega)$ is given by

(2.40) $$R_T(t)=\frac{1}{2T}f_T(t)-f_T(-t)=\frac{1}{2T}\int_{-T}^{T-t}f(t)f(t+\tau)\mathrm{d}\tau$$

for $t>0$ (similarly for $t<0$). Hence

(2.41) $$R_T(t)\to R(t),\ T\to\infty.$$

Helly's theorem then asserts that (2.36) is true. □

## 3. Fluctuations and musical scales

This section is still in the preliminary stage and is t be developed subsequently.

### 3.1. $1/f$ - fluctuations

Let $X(t)$ be a signal and consider its DFT

(3.1) $$\hat{X}(f)=\sum_{t=1}^{M}e^{-2\pi ift}X(t)$$

for the frequency $f$. In analogy with (2.13), its energy spectrum (power spectrum) is defined by

(3.2) $$E_X(f)=|\hat{X}(f)|^2$$

It was found (e.g. [15]) that many natural phenomena possess the energy spectrum depending on the frequency $1/f^{\beta}$. This kind of phenomena are called $1/f$-noise or **$1/f$ - fluctuation.**

In [18], the distribution of primes is considered from the point of view of $1/f$-fluctuation. However, in [7] there is criticism against the interpretation of power spectrum, saying that it applies to non-living. Indeed, any function having the Fourier transform is more or less thought of as linear because the Fourier integral shows that the original signal can be restored if we incorporate all possible frequencies (and amplitudes).

In studying music, which lies at this threshold of living and non-living, it would probably be reasonable to work with a more general class of functions which do not, in general, have Fourier transform, i.e. the class of finite power signals could be a possible target for research.

### 3.2. Musical scales

In statistics, the **strong law of large numbers** is well-known which claims that

(3.3) $$P\left\{\lim_{n\to\infty}\frac{r}{n}\right\}=1.$$

This means that the relative frequency $\frac{r}{n}$ of occurrences of an event $A$ tends to the true probability $p$ of the occurrences of $A$ with probability $1$.

In music the counterpart is **Pythagoras' law of small numbers**, which claims that only the small integer multiples of the fundamental notes can create harmony and consonance. It is also referred to as the law of cyclotomic numbers by Coxeter [2], see the remark at the end of this paper.

In the case of natural scales, they appear as $2^p3^q5^r$, where $q=-3,\ -2,\ -1,$ $0,\ 1,\ 2,\ 3$ and $r=-1, 0,\ 1.$

### 3.3. Various scales

The **pitch** of a **musical note** (hereafter abbreviated as a note) is defined by its frequency measured in **Herz** ($Hz$), cycles per second. The bigger the frequency, the higher the pitch.

In the case of Pythagorean scale, it is formed using only powers of $2$ and $3$. Two musical notes whose frequencies are different by powers of $2$ are thought of as belonging to the same equivalence class, or those which sound alike. Musicians express the equivalence relation by using the same label for those notes in the same class. E.g. if the middle $C$ has frequency $260$ Hz, then the note three octaves higher than middle $C$ has frequency $260\times2^3=2080$ Hz while the one two octaves below middle $C$ has frequency $260\times2^{-2}\approx65$ Hz. They are denoted by the same symbol $C$. As is suggested by the very name, one octave has $12$ **semitones**. Therefore, if we pile up the notes on the basic one, the $12$ th power is very important. In the case of the Pythagorean scale, what are piled up are powers of $\frac{3}{2}$, so that

(3.4) $$\left(\frac{3}{2}\right)^{12}\approx129.7.$$

which is a little higher than $7$ octaves: $2^7=128$. The interval

(3.5) $$\frac{129.7}{128}$$

is known as the **comma of Pythagoras**. This discrepancy accounts for many difficulties in obtaining an organized system of pitch.

In the equal tempered system, one octave is equally divided into $12$ semitones of ratio the $12$ th root of unity

(3.6) $$1\ \text{tempered semitone}=2^{1/12}=1.05946\cdots$$

### 3.4. The Pythagorean scale

The Pythagorean scale is made of $2^p3^q$, $p, q\in\mathbb{Z}, -3\leq q\leq3.$ As is explained above, starting from middle C, A (la) is obtained as $\left(\frac{3}{2}\right)^3\sim\frac{27}{16}.$

From middle C (doh) by piling up $\frac{3}{2}$, we get G (sol), then $\left(\frac{3}{2}\right)^2=\frac{9}{4}\sim\frac{9}{8}$, which is D (re). The next is $\left(\frac{3}{2}\right)^3=\frac{27}{8}\sim\frac{27}{16}$, which is A (la) with frequency 440 Hz. The 5th is $\left(\frac{3}{2}\right)^4=\frac{81}{16}\sim\frac{81}{64}=\left(\frac{3}{2}\right)^2$, which is E (mi).

The 6th is $\left(\frac{3}{2}\right)^2=\frac{243}{32}\sim\frac{243}{32}\frac{1}{2^2}=\frac{243}{128}$, which is B (si). Thus C (doh), G (sol), D (re), A (la), E (me), B (si). The 7th F (fa) is obtained by coming back from higher C. i.e. $2\cdot\left(\frac{3}{2}\right)^{-1}=\frac{4}{3}$.

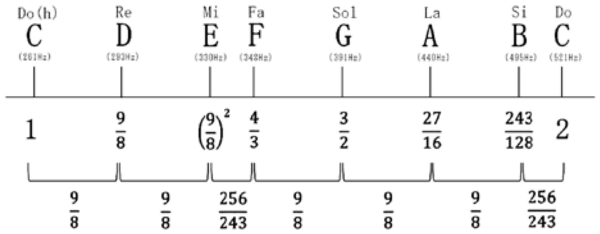


Figure 1. The Pythagorean Scale

This is the scale used by the Greeks and early medieval composers as the basis of the Ecclesiastic Mode. The scale is suitable for melodic writing but not satisfactory for harmonic writing (for modulation).

The home key could be any one of seven notes but what survived in tone-centered music nowadays are the **Ionian scale** beginning and ending on C and the **Aeolian scale** (with key-note A). The Ionian and Aeolian scales are known as the ordinary major and minor scales.

### 3.5. The Verdi pitch

The Verdi pitch is the scale which fixes the freq. of A to be 432.

**Theorem 3.1.** *The reason for the* $A=432$ *is that in the Pythagorean scale,*

(3.7) $$C=432\cdot\left(\frac{27}{16}\right)^{-1}=256=2^8$$

*while in just intonation* $C=426\frac{3}{2}\approx 427$ .

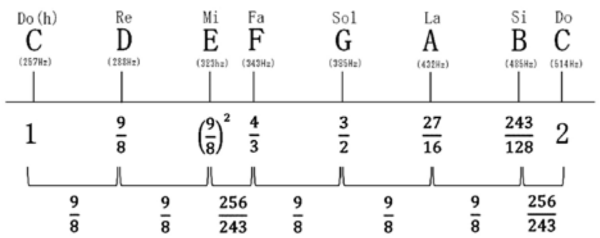


Figure 2. The Verdi pitch

### 3.6. The Helmholtz-Joachim scale and just intonation

Helmholtz with the help of the renowned violinist J. Joachim, made an experiment and tabulated the notes which are the most pleasing to the ears.

Table 1.1. Helmholtz-Joachim scale

| | $C$ | $E\flat$ | $E$ | $F$ | $G$ | $A\flat$ |
|---|---|---|---|---|---|---|
| int. | unison | minor 3rd | major 3rd | perfect 4th | perfect 5th | minor 6th |
| ratio | $\frac{1}{1}$ | $\frac{6}{5}$ | $\frac{5}{4}$ | $\frac{4}{3}$ | $\frac{3}{2}$ | $\frac{8}{5}$ |
| note | do | mi♭ | mi | fa | sol | la♭ |

Table 1.1. Helmholtz-Joachim scale (cont.)

| from $C$ | $A$ | $C$ |
|---|---|---|
| interval | major sixth | octave |
| ratio | $\frac{5}{3}$ | $\frac{2}{1}$ |
| note | la | do |

A partial explanation of Pythagoras' law of small numbers is given in [2] to the effect that sufficiently low notes have harmonics or overtones whose frequencies are exact integer multiples of the frequency of the original note.

Table 1.2. Multiples of low $C$

| mult. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| note | $C$ | $C$ | $G$ | $C$ | $E$ | $G$ | – | $C$ | $D$ | $E$ | – | $G$ |

Coxeter [2] states a speculation that the agreeable harmonics

(3.8) $$3,\ 4,\ 5,\ 6,\ 8,\ 10,\ 12$$

correspond to the number of sides of regular polygons that can be constructed by a ruler and a compass. A natural inverse question is where there are corresponding harmonics to $15$ and $17$ since $15$-gon was constructed by Euclid while $17$-gon was by Gauss in 1797. The interval $15$ from a low C to high B which thrilled the audience appears as the appoggiatura in the end of St. Matthew Passion. The interval $17$ were used by the remaining two of the $3$ B's, i.e. by Beethoven and Brahms. Since regular polygons inscribed in a circle divides it into equal parts, cyclotomy, Coxeter refers to the **law of cyclotomic numbers** rather than small numbers.

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