PAPER

Reduction Formulae in the Electric Circuit Equations and Their Applications

Naomi HARATANI [1)] Yoshimasa NAKANO [2)]

Abstract : After reviewing a wide class of electric circuit equation and its formal solution expressed independently of the fundamental tie-sets or fundamental cut-sets, we present some reduction formulae relating certain determinants and coefficients appearing in the solution. We also apply the formulae to generalizing the compensation theorems and to deriving a graph formula for the four-terminal constants.

1 Introduction

Although the formal solution of electric circuit equation and its general property have been widely studied, in many cases a solution is left being in a representation depending on a particular fundamental tie-sets or cut-sets; *i.e.*, each branch is classified as it belongs to the tree (T) or cotree ($\overline{T}$) of the circuit graph. However, without fundamental tie-sets or cut-sets, one cannot solve the equation systematically, such a solution is not convenient for one to treat all branches symmetrically, and it is complicated to develop further theoretical investigation or applications.

In this article, to avoid this kind of complexity, we first introduce a circuit equation taking all the branch voltages and currents as unknown quantities, and review a class of its solutions and a procedure how to recover the symmetry among the branches.

Hereafter, without loosing generality, we confine ourselves within the system of direct currents, and let n and b indicate the numbers of nodes and branches, respectively, in each connected graph of the circuit. Therefore, the numbers of the independent tie-sets and cut-sets are given by $l = b - n + 1$ and $m = n - 1$, respectively.

2 A Review on the Tree-independent Solution

Let V_i indicate the voltage and I_i the current of the i-th branch e_i in an electric circuit. However one tries to solve the circuit, one has to specify the set of unknown quantities among V_i and I_i. Here, all of V_i and I_i are treated as $2b$ unknown quantities and a circuit equation is written in the following form.

$$M \begin{pmatrix} V \\ \text{---} \\ I \end{pmatrix} = \begin{pmatrix} \tilde{\imath} \\ \text{---} \\ E \end{pmatrix} \qquad (M \in \mathbb{R}^{2b}_{2b};\ V, I, \tilde{\imath}, E \in \mathbb{R}^b) \tag{2.1}$$

where E is the electromotive force contained in e_i and $\tilde{\imath}_i$ is the current originated from external source current. (The present procedure should be that of the extreme case of *"the hybrid analysis"*[1].)

In order to solve the equation (2.1), we first choose a particular pair of tree (T) and cotree ($\overline{T}$) from the circuit graph G, and introduce fundamental cut-sets and fundamental tie-sets to specify the basis of

1) 近畿大学産業理工学部電気通信工学科　教授　haratani@fuk.kindai.ac.jp
2) 近畿大学産業理工学部電気通信工学科　教授　ynakano@fuk.kindai.ac.jp

the vector V and I. Then we obtain the explicit expressions of M and $\tilde{\iota}$ as follows:

$$M = \left(\begin{array}{c|c} G^{(\mathrm{d})}_{\overline{T}}\rho^{(\mathrm{C})} & 0 \\ 0 & \rho^{(\mathrm{D})} \\ \hline 1^{(b)} & R^{(\mathrm{d})} \end{array}\right) = \left(\begin{array}{cc|cc} G^{(\mathrm{d})}_{\overline{T}} & G^{(\mathrm{d})}_{\overline{T}}B & 0 & 0 \\ 0 & 0 & -{}^{t}B & 1^{(m)} \\ \hline 1^{(l)} & 0 & R^{(\mathrm{d})}_{\overline{T}} & 0 \\ 0 & 1^{(m)} & 0 & R^{(\mathrm{d})}_{T} \end{array}\right), \qquad \tilde{\iota} = \begin{pmatrix} 0 \\ \iota \end{pmatrix} \quad (\iota \in \mathbb{R}^{l}), \tag{2.2}$$

where $1^{(b)}$ is the b-dimensional unit matrix, $\rho^{(\mathrm{C})} = (1^{(l)}\ B) \in \mathbb{R}^{l}_{b}$ and $\rho^{(\mathrm{D})} = (-{}^{t}B\ 1^{(m)}) \in \mathbb{R}^{m}_{b}$ are fundamental tie-set matrix and fundamental cut-set matrix, $R^{(\mathrm{d})}$ is the branch resistance matrix, $R^{(\mathrm{d})}_{\overline{T}} \in \mathbb{R}^{l}_{l}$ is the sub-matrix of $R^{(\mathrm{d})}$ limited to the diagonal cotree space, and ι_p is the total current flowing over the p-th cut-set. (Hereafter, we omit definitions of self-evident notations.) Since M is regular[1], we can calculate the inverse matrix M^{-1} easily in terms of the fundamental resistance matrix $r = {}^{t}\rho^{(\mathrm{C})}R^{(\mathrm{d})}\rho^{(\mathrm{C})}$ and the fundamental conductance matrix $g = {}^{t}\rho^{(\mathrm{D})}G^{(\mathrm{d})}\rho^{(\mathrm{D})}$, that is,

$$M^{-1} = \left(\begin{array}{cc|cc} R^{(\mathrm{d})}_{\overline{T}} - Bg^{-1}{}^{t}B & Bg^{-1} & Bg^{-1}{}^{t}BG^{(\mathrm{d})}_{\overline{T}} & -Bg^{-1}G^{(\mathrm{d})}_{\overline{T}} \\ g^{-1}{}^{t}B & -g^{-1} & -g^{-1}{}^{t}BG^{(\mathrm{d})}_{\overline{T}} & g^{-1}G^{(\mathrm{d})}_{T} \\ \hline -r^{-1}R^{(\mathrm{d})}_{\overline{T}} & -G^{(\mathrm{d})}_{\overline{T}}Bg^{-1} & r^{-1} & r^{-1}B \\ -{}^{t}Br^{-1}R^{(\mathrm{d})}_{\overline{T}} & G^{(\mathrm{d})}_{T}g^{-1} & {}^{t}Br^{-1} & {}^{t}Br^{-1}B \end{array}\right). \tag{2.3}$$

Now, let us introduce another matrix $N \in \mathbb{R}^{2b}_{2b}$ defined by

$$N = \left(\begin{array}{c|c} -{}^{t}\rho^{(\mathrm{D})}g^{-1}\rho^{(\mathrm{D})} & {}^{t}\rho^{(\mathrm{D})}g^{-1}\rho^{(\mathrm{D})}G^{(\mathrm{d})} \\ \hline G^{(\mathrm{d})}{}^{t}\rho^{(\mathrm{D})}g^{-1}\rho^{(\mathrm{D})} & {}^{t}\rho^{(\mathrm{C})}r^{-1}\rho^{(\mathrm{C})} \end{array}\right) \tag{2.4}$$

$$= \left(\begin{array}{cc|cc} -Bg^{-1}{}^{t}B & Bg^{-1} & Bg^{-1}{}^{t}BG^{(\mathrm{d})}_{\overline{T}} & -Bg^{-1}G^{(\mathrm{d})}_{\overline{T}} \\ g^{-1}{}^{t}B & -g^{-1} & -g^{-1}{}^{t}BG^{(\mathrm{d})}_{\overline{T}} & g^{-1}G^{(\mathrm{d})}_{T} \\ \hline G^{(\mathrm{d})}_{\overline{T}}Bg^{-1}{}^{t}B & -G^{(\mathrm{d})}_{\overline{T}}Bg^{-1} & r^{-1} & r^{-1}B \\ -G^{(\mathrm{d})}_{T}g^{-1}{}^{t}B & G^{(\mathrm{d})}_{T}g^{-1} & {}^{t}Br^{-1} & {}^{t}Br^{-1}B \end{array}\right), \tag{2.5}$$

to compare with M^{-1}. Since the matrices M^{-1} and N have the same elements except the first l columns[2],

[1] $|M| = |g|\,|R^{(\mathrm{d})}_{T}| \neq 0$

[2] $G^{(\mathrm{d})}_{\overline{T}}Bg^{-1}{}^{t}B + r^{-1}R^{(\mathrm{d})}_{\overline{T}} = 1^{(l)}$, $G^{(\mathrm{d})}_{T}g^{-1}{}^{t}B = {}^{t}Br^{-1}R^{(\mathrm{d})}_{\overline{T}}$.

we can write the solution of (2.1) in terms of N instead of M^{-1}, as follows:

$$\begin{pmatrix} V \\ \hline I \end{pmatrix} = N \begin{pmatrix} \tilde{\iota} \\ \hline E \end{pmatrix} , \quad \text{with} \quad N = \left(\begin{array}{c|c} -\dfrac{h}{|g|} & \dfrac{hG^{(\mathrm{d})}}{|g|} \\ \hline \dfrac{G^{(\mathrm{d})}h}{|g|} & \dfrac{f}{|r|} \end{array}\right) , \tag{2.6}$$

where

$$f = {}^t\!\rho^{\mathrm{C}}\, {}^t\!\Delta^{(r)} \rho^{(\mathrm{C})} , \quad h = {}^t\!\rho^{\mathrm{D}}\, {}^t\!\Delta^{(g)} \rho^{(\mathrm{D})} . \tag{2.7}$$

(The symbols $\Delta^{(r)}$ and $\Delta^{(g)}$ denote the cofactor matrices of r and g, respectively.)

The above representation for N shows that it is independent of the fundamental tie-sets and the fundamental cut-sets[2], and that the solution (2.6) is invariant under the transformation of $\tilde{\iota}$ such as

$$\tilde{\iota} \to \tilde{\iota} + {}^t\!\rho^{(\mathrm{C})} J \qquad (\text{with } J \in \mathbb{R}^l \text{ being arbitrary}) . \tag{2.8}$$

This means that the expression of the solution (2.6) allows the external and internal currents forking arbitrarily as long as the current conservation is maintained. Therefore, the expression, which is written in terms of $\tilde{\iota}$ replaced as (2.8), is independent of a particular set of T and $\overline{T}$.

It should be insisted that one cannot write a symmetric solution without arbitrary variables such as J; otherwise, one has to sacrifice the symmetry among the branches, where the current ι has no ambiguity as

$$\iota = \rho^{(\mathrm{D})} \tilde{\iota} \to \rho^{(\mathrm{D})} (\tilde{\iota} + {}^t\!\rho^{(\mathrm{C})} J) = \iota . \tag{2.9}$$

There are several advantages in writing the solutions for electric circuits using $|r|$, f, $|g|$ and h, as follows: (i) Each of the quantities has a possible expression independent of a set of T and $\overline{T}$ as already stated above. (ii) There are some remarkable relations among them; *e.g.*,

$$\frac{R^{(\mathrm{d})} f}{|r|} + \frac{hG^{(\mathrm{d})}}{|g|} = \frac{fR^{(\mathrm{d})}}{|r|} + \frac{G^{(\mathrm{d})}h}{|g|} = 1^{(b)} \tag{2.10}$$

(iii) They are calculated by means of formulae related with the topology of circuit graphs[3]. (iv) Their differential coefficients w.r.t. R_k and G_k are written by certain combinations of themselves, which will be shown as reduction formulae.

3 Reduction Formulae

The differential coefficients of the determinant of fundamental resistance matrix $|r|$ and of the matrix elements of f w.r.t. a resistance (say, R_1) are calculated as,

$$\frac{\partial |r|}{\partial R_1} = \frac{\partial r_{\alpha\beta}}{\partial R_1} \Delta^{(r)}_{\alpha\beta} = f_{11} = |r^-| , \tag{3.1}$$

$$\frac{\partial f_{ij}}{\partial R_1} = \frac{\partial}{\partial R_1} (|r|\, {}^t\!\rho^{(\mathrm{C})} r^{-1} \rho^{(\mathrm{C})}) = \frac{1}{|r|} (f_{11} f_{ij} - f_{i1} f_{1j}) , \tag{3.2}$$

[3]See the reference [3] for Eq.(2.10) and many of related formulae.

where $|r^-|$ denotes the determinant of the fundamental resistance matrix of the circuit $G^- = G - e_1$[4]. Then, in general for $s < l$, $\partial^s|r|/\partial R_{\tau(1)} \cdots \partial R_{\tau(s)}$ coincides with the determinant of the fundamental matrix of $G - \{e_{\tau(1)}, e_{\tau(2)}, \dots, e_{\tau(s)}\}$.

In the following, we present some theorems which are applicable to arbitrary sets of $\{e_{\tau(1)}, \dots, e_{\tau(s)}\}$, by generalizing the relations (3.1) and (3.2). For this purpose, we present a lemma.

Lemma 3.1 Let σ and τ be mappings from $\{1, \dots, \nu, \nu+1\}$ $(1 \le \nu \le b-1)$ to $\{1, \dots, b\}$ and $\sigma(\nu+1) = \tau(\nu+1)$, then

$$\frac{\partial}{\partial R_{\sigma(\nu+1)}} \frac{|f^{(\nu)}|}{|r|^{\nu-1}} = \frac{|f^{(\nu+1)}|}{|r|^{\nu}} \quad \text{with} \quad f^{(\nu)} = \begin{pmatrix} f_{\sigma(1)\tau(1)} & \cdots & f_{\sigma(1)\tau(\nu)} \\ \vdots & \ddots & \vdots \\ f_{\sigma(\nu)\tau(1)} & \cdots & f_{\sigma(\nu)\tau(\nu)} \end{pmatrix} . \tag{3.3}$$

Proof:

$$\frac{\partial}{\partial R_{\sigma(\nu+1)}} \frac{|f^{(\nu)}|}{|r|^{\nu-1}} = -\frac{\nu-1}{|r|^{\nu}} \frac{\partial|r|}{\partial R_{\sigma(\nu+1)}} |f^{(\nu)}| + \frac{1}{|r|^{\nu-1}} \frac{\partial|f^{(\nu)}|}{\partial R_{\sigma(\nu+1)}} . \tag{3.4}$$

Taking a convention that $\nu + 1 = \nu'$ for a while, each term in Eq.(3.4) can be rewritten as

$$\frac{\partial|r|}{\partial R_{\sigma(\nu+1)}} = \frac{\partial|r|}{\partial R_{\sigma(\nu')}} = f_{\sigma(\nu')\sigma(\nu')} = f_{\sigma(\nu')\tau(\nu')} , \tag{3.5}$$

$$\frac{\partial|f^{(\nu)}|}{\partial R_{\sigma(\nu+1)}} = \frac{\partial|f^{(\nu)}|}{\partial R_{\sigma(\nu')}} = \sum_{i=1}^{\nu}\sum_{j=1}^{\nu} \frac{\partial f_{\sigma(i)\tau(j)}}{\partial R_{\sigma(\nu')}} \Delta_{ij}^{(\nu)} \tag{3.6}$$

$$= \sum_{i=1}^{\nu}\sum_{j=1}^{\nu} \frac{f_{\sigma(\nu')\sigma(\nu')} f_{\sigma(i)\tau(j)} - f_{\sigma(i)\sigma(\nu')} f_{\sigma(\nu')\tau(j)}}{|r|} \Delta_{ij}^{(\nu)} \tag{3.7}$$

$$= \sum_{i=1}^{\nu}\sum_{j=1}^{\nu} \frac{f_{\sigma(\nu')\tau(\nu')} f_{\sigma(i)\tau(j)} - f_{\sigma(i)\tau(\nu')} f_{\sigma(\nu')\tau(j)}}{|r|} \Delta_{ij}^{(\nu)} \tag{3.8}$$

$$= \frac{1}{|r|} \left(\nu f_{\sigma(\nu')\tau(\nu')} |f^{(\nu)}| - \sum_{i=1}^{\nu}\sum_{j=1}^{\nu} f_{\sigma(i)\tau(\nu')} f_{\sigma(\nu')\tau(j)} \Delta_{ij}^{(\nu)} \right) , \tag{3.9}$$

with $\Delta_{ij}^{(\nu)}$ being the (i,j)-cofactor of $f^{(\nu)}$. From these relation, Eq.(3.4) can be rewritten as

$$\frac{\partial}{\partial R_{\sigma(\nu+1)}} \frac{|f^{(\nu)}|}{|r|^{\nu-1}} = \frac{1}{|r|^{\nu}} \left(f_{\sigma(\nu')\tau(\nu')} |f^{(\nu)}| - \sum_{i=1}^{\nu}\sum_{j=1}^{\nu} f_{\sigma(i)\tau(\nu')} f_{\sigma(\nu')\tau(j)} \Delta_{ij}^{(\nu)} \right) . . \tag{3.10}$$

On the other hand, introducing $\mathcal{D}'_{\kappa\lambda}$ and $\mathcal{D}_{\kappa\lambda}$ as the matrices $f^{(\nu+1)}$ and $f^{(\nu)}$, respectively, with the κ-th row and the λ-th column removed, the determinant $|f^{(\nu+1)}|$ is expanded as

$$|f^{(\nu+1)}| = \sum_{i=1}^{\nu+1} f_{\sigma(i)\tau(\nu')} \Delta_{i\nu'}^{(\nu')} = \sum_{i=1}^{\nu+1} (-1)^{i+\nu'} f_{\sigma(i)\tau(\nu')} |\mathcal{D}'_{i\nu'}| \tag{3.11}$$

[4] Contrary to $|r^-|$, $|g^-| = |g|_{G_1=0}$.

$$= \sum_{i=1}^{\nu}(-1)^{i+\nu'} f_{\sigma(i)\tau(\nu')}|\mathcal{D}'_{i\nu'}| + f_{\sigma(\nu')\tau(\nu')}(-1)^{\nu'+\nu'}|\mathcal{D}'_{\nu'\nu'}| \tag{3.12}$$

$$= \sum_{i=1}^{\nu}(-1)^{i+\nu'} f_{\sigma(i)\tau(\nu')} \sum_{j=1}^{\nu}(-1)^{\nu'-1+j} f_{\sigma(\nu')\tau(j)}|\mathcal{D}_{ij}| + f_{\sigma(\nu')\tau(\nu')}|f^{(\nu)}| \tag{3.13}$$

$$= -\sum_{i=1}^{\nu}\sum_{j=1}^{\nu} f_{\sigma(i)\tau(\nu')} f_{\sigma(\nu')\tau(j)} \Delta_{ij}^{(\nu)} + f_{\sigma(\nu')\tau(\nu')}|f^{(\nu)}| \,. \tag{3.14}$$

This is exactly the same as the factor in the r.h.s. of Eq.(3.10), which immediately proves Eq.(3.3). ■

The following theorems are generalizations of (3.1) and (3.2).

Theorem 3.1 Let τ be a mapping from $\{1, \ldots, s\}$ $(1 \leq s \leq b)$ to $\{1, \ldots, b\}$, then

$$\frac{\partial^s |r|}{\partial R_{\tau(s)} \cdots \partial R_{\tau(1)}} = \frac{1}{|r|^{s-1}} \begin{vmatrix} f_{\tau(1)\tau(1)} & \cdots & f_{\tau(1)\tau(s)} \\ \vdots & \ddots & \vdots \\ f_{\tau(s)\tau(1)} & \cdots & f_{\tau(s)\tau(s)} \end{vmatrix} \,. \tag{3.15}$$

□

Theorem 3.2 Let τ be a mapping from $\{1, \ldots, s\}$ $(1 \leq s \leq b)$ to $\{1, \ldots, b\}$, then

$$\frac{\partial^s f_{ij}}{\partial R_{\tau(s)} \cdots \partial R_{\tau(1)}} = \frac{1}{|r|^s} \left|\begin{array}{c|ccc} f_{ij} & f_{i\tau(1)} & \cdots & f_{i\tau(s)} \\ \hline f_{\tau(1)j} & f_{\tau(1)\tau(1)} & \cdots & f_{\tau(1)\tau(s)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{\tau(s)j} & f_{\tau(s)\tau(1)} & \cdots & f_{\tau(s)\tau(s)} \end{array}\right| \,. \tag{3.16}$$

□

Since these theorems are easily proved by means of the mathematical induction, we here omit the proofs.

We call Eqs. (3.15) and (3.16) reduction formulae in sense that the differential coefficient of $|r|$ and f_{ij} are expressed in terms of themselves.

Similarly to $|r|$ and f, we can derive another set of reduction formulae for $|g|$ and h, which can be written down simply by replacing $|r|$ and f_{ij} with $|g|$ and h_{ij}, respectively.

4 Applications

4.1 Response Formulae

Taking advantage of the reduction formulae, we can derive a type of response formulae corresponding to the changes of $R^{(\mathrm{d})}$ and $G^{(\mathrm{d})}$. Actually, we have relations between higher rank derivatives as

$$\Delta R_{i_1} \cdots \Delta R_{i_k} \frac{\partial^k}{\partial R_{i_1} \cdots \partial R_{i_k}} \frac{f}{|r|} = (-1)^k \, k! \, \frac{f(\Delta R^{(\mathrm{d})} f)^k}{|r|^{k+1}} \,, \tag{4.1}$$

$$\Delta G_{i_1} \cdots \Delta G_{i_k} \frac{\partial^k}{\partial G_{i_1} \cdots \partial G_{i_k}} \frac{h}{|g|} = (-1)^k \, k! \, \frac{h(\Delta G^{(\mathrm{d})} h)^k}{|g|^{k+1}} \,. \tag{4.2}$$

(We can easily prove these by the mathematical induction.) Then we immediately obtain the following changes as response formulae.

$$\frac{f}{|r|} \to \frac{f}{|r|}\left(1^{(b)} + \frac{\Delta R^{(\mathrm{d})} f}{|r|}\right)^{-1} = \left(1^{(b)} + \frac{f \Delta R^{(\mathrm{d})}}{|r|}\right)^{-1} \frac{f}{|r|} \ , \tag{4.3}$$

$$\frac{h}{|g|} \to \frac{h}{|g|}\left(1^{(b)} + \frac{\Delta G^{(\mathrm{d})} h}{|g|}\right)^{-1} = \left(1^{(b)} + \frac{h \Delta G^{(\mathrm{d})}}{|g|}\right)^{-1} \frac{h}{|g|} \ . \tag{4.4}$$

4.2 Compensation Theorems Revisited

We can generalize so-called compensation theorems making use of our response formulae. We start with the solution for the branch currents in a circuit G, that is,

$$I = \frac{f}{|r|}(E - R^{(\mathrm{d})}\tilde{\imath}) + \tilde{\imath} \tag{4.5}$$

In a change of the branch resistances as $R \to R' = R + \Delta R$, the branch currents are changed as

$$I \to I' = \left(1^{(b)} + \frac{f \Delta R^{(\mathrm{d})}}{|r|}\right)^{-1} \frac{f}{|r|}[E - (R^{(\mathrm{d})} + \Delta R^{(\mathrm{d})})\,\tilde{\imath}\,] + \tilde{\imath} \ . \tag{4.6}$$

The present change is recovered by a suitable change of the electromotive forces as $E \to E' = E + \Delta E$. Since the recovery condition $I' \to I$ is given by

$$I' \to \left(1^{(b)} + \frac{f \Delta R^{(\mathrm{d})}}{|r|}\right)^{-1} \frac{f}{|r|}[E + \Delta E - (R^{(\mathrm{d})} + \Delta R^{(\mathrm{d})})\,\tilde{\imath}\,] + \tilde{\imath} = I \ , \tag{4.7}$$

we obtain the following equations successively.

$$\frac{f}{|r|}[E + \Delta E - (R^{(\mathrm{d})} + \Delta R^{(\mathrm{d})})\,\tilde{\imath}\,] = \frac{f}{|r|}\left(1^{(b)} + \frac{\Delta R^{(\mathrm{d})} f}{|r|}\right)(E - R^{(\mathrm{d})}\,\tilde{\imath}\,) \ , \tag{4.8}$$

$$\frac{f}{|r|}\Delta E = \frac{f}{|r|}\Delta R^{(\mathrm{d})}\left[\frac{f}{|r|}(E - R^{(\mathrm{d})}\tilde{\imath}) + \tilde{\imath}\right] = \frac{f}{|r|}\Delta R^{(\mathrm{d})} I \ . \tag{4.9}$$

Then we finally find the suitable change ΔE as

$$\Delta E = \Delta R^{(\mathrm{d})} I + {}^{t}\!\rho^{(D)} \Delta K \qquad (\text{with } \Delta K \in \mathbb{R}^m \text{ being arbitrary}) \ . \tag{4.10}$$

Note that the second term at the r.h.s. of (4.10) keeps the branch currents I invariant even if ΔR are absent.

Meanwhile, the change of the branch currents by the independent change of the branch resistances is given by

$$\begin{aligned}
\Delta I &= I' - I && (4.11)\\
&= \frac{h}{|r|}\left(1^{(b)} + \frac{\Delta R^{(\mathrm{d})} f}{|r|}\right)^{-1}\left[E - (R^{(\mathrm{d})} + \Delta R^{(\mathrm{d})})\,\tilde{\imath} - \left(1^{(b)} + \frac{\Delta R^{(\mathrm{d})} f}{|r|}\right)(E - R^{(\mathrm{d})}\tilde{\imath})\right] && (4.12)\\
&= \frac{h}{|r|}\left(1^{(b)} + \frac{\Delta R^{(\mathrm{d})} f}{|r|}\right)^{-1}(-\Delta R^{(\mathrm{d})} I) = \frac{h}{|r|}\left(1^{(b)} + \frac{\Delta R^{(\mathrm{d})} f}{|r|}\right)^{-1}(-\Delta E) && (4.13)\\
&= \frac{f}{|r|}\bigg|_{R \to R + \Delta R}(E - R^{(\mathrm{d})}\tilde{\imath})\bigg|_{E \to -\Delta E,\ \tilde{\imath} \to 0} = I\bigg|_{R \to R + \Delta R,\ E \to -\Delta E,\ \tilde{\imath} \to 0} \ , && (4.14)
\end{aligned}$$

as expected.

In a similar way, since the solution of the cut-set equation for the branch voltages of a circuit G is given by

$$V = \frac{h}{|g|}(G^{(\mathrm{d})}E - \tilde{\iota}) \ , \tag{4.15}$$

the voltages are changed as

$$V \to V' = \left(1^{(b)} + \frac{h\Delta G^{(\mathrm{d})}}{|g|}\right)^{-1} \frac{h}{|g|}[(G^{(\mathrm{d})} + \Delta G^{(\mathrm{d})})E - \tilde{\iota}] \tag{4.16}$$

under the change of the branch conductances, $G \to G' = G + \Delta G$.

This time, the voltages V' are recovered to V by the change of the external source currents as $\iota \to \iota' = \iota + \Delta\iota$. This is realized by

$$V' \to \left(1^{(b)} + \frac{h\Delta G^{(\mathrm{d})}}{|g|}\right)^{-1} \frac{h}{|g|}[(G^{(\mathrm{d})} + \Delta G^{(\mathrm{d})})E - (\tilde{\iota} + \Delta\tilde{\iota})] = V \ , \tag{4.17}$$

which is led to the following equations.

$$\frac{h}{|g|}[(G^{(\mathrm{d})} + \Delta G^{(\mathrm{d})})E - (\tilde{\iota} + \Delta\tilde{\iota})] = \frac{h}{|g|}\left(1^{(b)} + \frac{\Delta G^{(\mathrm{d})}h}{|g|}\right)(G^{(\mathrm{d})}E - \tilde{\iota}) \ , \tag{4.18}$$

$$\frac{h}{|g|}\Delta\tilde{\iota} = \frac{h}{|g|}\Delta G^{(\mathrm{d})}\left[E - \frac{h}{|g|}(G^{(\mathrm{d})}E - \tilde{\iota})\right] = \frac{h}{|g|}\Delta G^{(\mathrm{d})}(E - V) \ . \tag{4.19}$$

Then we again a similar result, which is

$$\Delta\tilde{\iota} = \Delta G^{(\mathrm{d})}(E - V) + {}^t\!\rho^{(\mathrm{C})}\Delta J \qquad (\text{with } \Delta J \in \mathbb{R}^l \text{ being arbitrary}) \ . \tag{4.20}$$

Contrary to $\Delta\tilde{\iota}$, the currents $\Delta\iota$ have no ambiguities because

$$\Delta\iota = \rho^{(\mathrm{D})}\Delta\tilde{\iota} = \rho^{(\mathrm{D})}\Delta G^{(\mathrm{d})}(E - V) \ . \tag{4.21}$$

The change of the branch voltages under the independent change of the conductances is similarly calculated as

$$\Delta V = V' - V \tag{4.22}$$

$$= \frac{h}{|g|}\left(1^{(b)} + \frac{\Delta G^{(\mathrm{d})}h}{|g|}\right)^{-1}\left[(G^{(\mathrm{d})} + \Delta G^{(\mathrm{d})})E - \tilde{\iota} - \left(1^{(b)} + \frac{\Delta G^{(\mathrm{d})}h}{|g|}\right)(G^{(\mathrm{d})}E - \tilde{\iota})\right] \tag{4.23}$$

$$= \frac{h}{|g|}\left(1^{(b)} + \frac{\Delta G^{(\mathrm{d})}h}{|g|}\right)^{-1}\Delta G^{(\mathrm{d})}(E - V) = \frac{h}{|g|}\left(1^{(b)} + \frac{\Delta G^{(\mathrm{d})}h}{|g|}\right)^{-1}(\Delta\tilde{\iota} - {}^t\!\rho^{(\mathrm{C})}\Delta J) \tag{4.24}$$

$$= \frac{h}{|g|}\left(1^{(b)} + \frac{\Delta G^{(\mathrm{d})}h}{|g|}\right)^{-1}\Delta\tilde{\iota} \tag{4.25}$$

$$= \left.\frac{h}{|g|}\right|_{G \to G+\Delta G}\left[(G^{(\mathrm{d})} + \Delta G^{(\mathrm{d})})E - \tilde{\iota}\right]_{E \to 0,\ \tilde{\iota} \to -\Delta\tilde{\iota}} = V\Big|_{G \to G+\Delta G,\ E \to 0,\ \tilde{\iota} \to -\Delta\tilde{\iota}} \ . \tag{4.26}$$

4.3 A Graph Formula for the Four-terminal Circuit

Let G be the graph of a four-terminal circuit. By adding new branches $e_{\dot{1}}$ and/or $e_{\dot{2}}$ connecting the pair of input/output terminals, we introduce the following graphs.

$$G^{+-} = G \cup e_{\dot{1}} \ , \quad G^{-+} = G \cup e_{\dot{2}} \ , \quad G^{++} = G \cup e_{\dot{1}} \cup e_{\dot{2}} \qquad (4.27)$$

We also express their fundamental resistance matrices as $|r^{+-}|$, $|r^{-+}|$ and $|r^{++}|$.

Since the solution of the tie-set equation for the circuit G^{++} with no external source currents contains

$$I_{\dot{1}} = \frac{f^{++}_{\dot{1}\dot{1}} V_{\dot{1}} - f^{++}_{\dot{1}\dot{2}} V_{\dot{2}}}{|r^{++}|} \ , \qquad I_{\dot{2}} = \frac{f^{++}_{\dot{2}\dot{1}} V_{\dot{1}} - f^{++}_{\dot{2}\dot{2}} V_{\dot{2}}}{|r^{++}|} \ , \qquad (R_{\dot{1}} = R_{\dot{2}} = 0) \ , \qquad (4.28)$$

in the input/output sector, we can rearrange the partial solution (4.28) as

$$\begin{pmatrix} V_{\dot{1}} \\ I_{\dot{1}} \end{pmatrix} = \frac{1}{f^{++}_{\dot{2}\dot{1}}} \begin{pmatrix} f^{++}_{\dot{2}\dot{2}} & |r^{++}| \\ \dfrac{f^{++}_{\dot{1}\dot{1}} f^{++}_{\dot{2}\dot{2}} - f^{++}_{\dot{2}\dot{1}} f^{++}_{\dot{1}\dot{2}}}{|r^{++}|} & f^{++}_{\dot{1}\dot{1}} \end{pmatrix} \begin{pmatrix} V_{\dot{2}} \\ I_{\dot{2}} \end{pmatrix} . \qquad (4.29)$$

Furthermore, making use of the reduction formulae and relations between different graphs such as

$$\frac{1}{|r^{++}|} \begin{vmatrix} f^{++}_{\dot{1}\dot{1}} & f^{++}_{\dot{1}\dot{2}} \\ f^{++}_{\dot{2}\dot{1}} & f^{++}_{\dot{2}\dot{2}} \end{vmatrix} = \frac{\partial^2 |r^{++}|}{\partial R_{\dot{2}} \partial R_{\dot{1}}} = |r| \ , \qquad f^{++}_{\dot{2}\dot{2}} = |r^{+-}| \ , \qquad f^{++}_{\dot{1}\dot{1}} = |r^{-+}| \ , \qquad (4.30)$$

we can rewrite the r.h.s. of (4.29) and finally obtain that

$$F = \frac{1}{f^{++}_{\dot{2}\dot{1}}} \begin{pmatrix} |r^{+-}| & |r^{++}| \\ |r| & |r^{-+}| \end{pmatrix} \qquad (R_{\dot{1}} = R_{\dot{2}} = 0) \qquad (4.31)$$

for the four-terminal constants F.

It is possible to evaluate the all factors at the r.h.s. of (4.31) using the following formulae[3].

$$|r| = \sum_{\bar{T}} \prod_{e_k \in \bar{T}} R_k \ , \qquad f_{ij} = \sum_{C} \begin{bmatrix} C \\ e_i \end{bmatrix} \begin{bmatrix} C \\ e_j \end{bmatrix} \sum_{T^0 \supset C} \prod_{e_k \in \overline{T^0}} R_k \ . \qquad (4.32)$$

(The symbols C and T^0 denote a tie-set and a pseudotree, respectively, in G, and $\begin{bmatrix} C \\ e \end{bmatrix}$ means the tie-set index, with which the elements of tie-set matrix are written as $\rho^{(C)}_{\alpha\, i} = \begin{bmatrix} C_\alpha \\ e_i \end{bmatrix}$.) Therefore, Eq.(4.31) is nothing but the graph formula to calculate the matrix elements of F.

5 Summary

Taking advantage of the ambiguity originated from the external source currents, we obtained an expression of the solution for a wide class of electric circuit equations which is independent of a particular set of tree and cotree of the graph. We also obtained certain reduction formulae, and applied them to the compensation theorems and to a graph formula for the four-terminal constants. We found that the present formulae prove extremely powerful in symmetric calculations and in relating the different types of circuit graphs.

References

[1] M. Takamatsu and S. Iwata, *"Index Characterization of Differential-Algebraic Equations in Hybrid Analysis for Circuit Simulation"*, METR 2008-10, Dep. Math. Info. and Tech., Univ. of Tokyo, 2008.

[2] C. A. Desoer and E. S. Kuh, *"Basic Circuit Theory"*, McGraw-Hills 1969.

[3] N. Nakanishi, Prog. Theor. Phys. **17**, 401 – 418, 1957; N. Nakanishi, *"Quantum Field Theory"*, Baihukan, 1975.