PAPER

# Record of the second and third interdisciplinary seminars

K.TAKAHASHI[1) ] G.HIRANO[2)] T.KAIDA[3)] S.KANEMITSU[4)] H.TSUKADA[5)] T.MATSUZAKI[6)]

概要: 著者らは2010年11月から月に1度のペースで定期的に学際的セミナーを開催している．本セミナーでは文献を批判的に精読し，工学領域における数学的基盤の見直しなどを試みることにより，最終的には大学教育における新たな教科書を作成することを目的としている．これまで本セミナーを5回開催し，実際の論文のレフェリーレポートを基にしたクリティカルリーディングに始まり，自動制御システム→マン・マシンシステム→離散フーリエ変換→符号理論の順に底奏通音としての波動に乗って議論してきた．本稿はこれまでの本セミナーのうち第2回と第3回における内容の概略をまとめたものである．

キーワード：太制御理論，マン・マシンシステム，離散フーリエ変換，符号理論
Keywords: Control Theory, Man-Machine System, Discrete Fourier Transform, Coding Theory

## 1. Introduction

The following is the record of four conducted interdisciplinary seminars whose main body is a brushed-up version of the material of the second and third ones. The first seminar was centered around how to make critical reading and a referee report was presented as an illustration in which the fractional calculus was introduced. Fractional calculus can provide traditional PIDs with a novel and higher performance − FOPID (Fractional Order Proportional-Integral-Differential) controllers − at the sacrifice of increased complexities arising from specifications of the 5 parameters including integral and derivative orders. An optimization method can be used for designing it which make use of PSO (Particle Swarm Optimization), a developed form of genetic algorithm.

This notion has been pursued in the second seminar whose main purpose was to give an outline of one of our projects toward the **control of linear systems**. The scope, however, is too wide and diverse to be stated in a few seminars, and we restricted to a possible generalization of various control methods to a chain scattering representation framework and their accommodation in chain scattering representation approach to $H^{\infty}$-control problem[Kim].

We may view the unit feedback system as one of typical examples of the chain scattering representation, which gives a better insight into the structure of the system, so can be the FOPID controllers.

The control of (quasi-) linear systems has been analytically continued to the “man-machine system" in human factors in the third seminar, in which our main concern is the prevention of noise pollution. As a preliminary to this, we concentrated on the Weber-Fechner law and completed a research paper [Tak]. We also introduced the notion of Fourier transform (since the sound is a wave and Fourier analysis naturally comes onto stage) whose discrete version has been one of the main ingredients of the fourth seminar devoted to coding theory.

Thus we are led by the following loose chain of notions.

- control theory → man-machine system (noise pollution) → (discrete) Fourier transform → coding theory

## 2. Aims and scope

The aim of this seminar is two-fold. On one hand, we try to lay a sound mathematical foundation of the underlying principles in engineering disciplines, restricted, however, to those which are the majors of the main members, i.e. mechanical engineering, human factors (ergonomics), coding theory, scheduling and basic sciences. On the other hand, we

1) 近畿大学産業理工学部情報学科　講師　ktakahas@fuk.kindai.ac.jp
2) 近畿大学産業理工学部電気通信工学科　講師　hira@fuk.kindai.ac.jp
3) 近畿大学産業理工学部情報学科　准教授　kaida@fuk.kindai.ac.jp
4) 近畿大学産業理工学部情報学科　教授　kanemitu@fuk.kindai.ac.jp
5) 近畿大学産業理工学部情報学科　教授　tsukada@fuk.kindai.ac.jp
6) 近畿大学産業理工学部電気通信工学科　講師　takanori@fuk.kindai.ac.jp

try to manifest the passage that research and education are two sides of a coin, i.e. we shall illustrate by our seminars what research should be and how it should affect education rather than teaching. Here we have used the word "education" to mean deriving the hidden potential talent of each enthusiastic student **with enough basic**, and not mere teaching as a university teacher. We shall try to be educators rather than teachers and try to encourage the younger generation to develop properly their potential qualifications.

In the Japanese society, students are forced to choose their major before they can think very seriously about it and they just choose one according to the fashion at the time, which usually turns out to be unsuccessful. However, with enough basic, every student can, whether in humane or scientific subjects, keep working on them and eventually can master part of the subjects.

We have found that the existing textbooks in Japan on engineering disciplines are rather poor compared with those in English. One of the main reasons is that the foundation is not very proper and there is no mention of the important definition of certain notions or terminologies and above all things, there are too many unnecessary classifications and labels. We may try to improve the presentation of these existing textbooks to prepare a new 21st century fox type textbook.

The first principle is **not to classify subjects** and try all that looks interesting. We must keep in mind the passage "It is a curious fact that radical conceptual changes in a science is often started by those who have not received their initial professional training in that science." Here we may replace "science" by a subject in engineering. The Japanese nation is very fond of classifying and labeling probably because they would feel comfortable by doing so and forget those which are not their majors. However, we're now living in the time of global standards. And we should pay attention to the research style of scientists in Europe and US, who can change their subjects quite easily and can pursue new disciplines as their curiosity lead them. We must adopt this frontier spirit and should be interested in all that is interesting. We are not to exclude other disciplines even they look irrelevant to our own subjects.

There is the principle of **serendipity** to the effect that if you look for something significant very seriously, you could find another thing which is as important as what you have been looking for. We expect this will manifest in the research of each member and would lead to a full paper of type $A$.

We shall gather once a month, give some talks if possible about topics in which any one of the members is interested. Or else graduate students can give seminar talks. If a foreign visitor comes, we may organize a seminar surrounding him.

We pursue **intelligent enjoyment** as well as **originality** as you can see in the book "Davinci decoded" by Gelb or as in "Le E'sprit" by Le Corbusier.

Wirmüssenwissen.Wirwerdenwissen.

We also work on the principle of **noblesse oblige**– nobles' obligations, meaning that we should make devoted efforts toward the education of motivated enthusiastic students at the sacrifice of worse students' evaluations than incompetent ones who would give easier and surfacial courses. But this kind of teaching which possibly occupy the majority of university teaching in Japan would create a great number of incompetent sciolists.

Only the spirit, when it blows upon a clay, can create man.

Even if our evaluations are funnily low, we should be competitive, genuine educators, because that is our "raison d'être", which is the other side of the coin.

## 3. Chain scattering representation

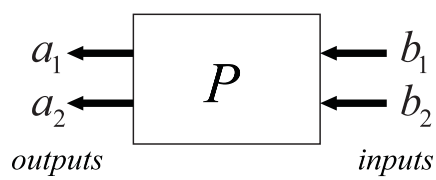


Figure 1.1

The following is an incorporation of [Kim, p. 7, p. 67, p. 94]. Figs 1.1 and 7.1 in [Kim] look contradictory. Suppose $\boldsymbol{a}_1 \in \mathbb{R}^m$, $\boldsymbol{a}_2 \in \mathbb{R}^q$, $\boldsymbol{b}_1 \in \mathbb{R}^r$ and $\boldsymbol{b}_2 \in \mathbb{R}^p$ are related by

(3.1) $$\begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \end{pmatrix} = P \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix},$$

where

(3.2) $$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix},$$

where $P$ is an $(r+p)\times(m+q)$-matrix which should not be confused with the plan and also that $\mathbf{a}_2$ is fed back to $\mathbf{b}_2$ by

(3.3) $$\boldsymbol{b}_2 = K\boldsymbol{a}_2,$$

where $K$ is a controller, which is often denoted by $C$. Then the **closed-loop transfer function** defined by

(3.4) $$\boldsymbol{a}_1 = \Phi \boldsymbol{b}_1$$

is given by

(3.5) $$\Phi = P_{11} + P_{12}(E - P_{22}K)^{-1}KP_{21}.$$

For (3.1) means that

(3.6) $$\begin{aligned}\boldsymbol{a}_1 &= P_{11}\boldsymbol{b}_1 + P_{12}\boldsymbol{b}_2,\\ \boldsymbol{a}_2 &= P_{21}\boldsymbol{b}_1 + P_{22}\boldsymbol{b}_2.\end{aligned}$$

Multiplying the second equality by $K$ and incorporating (3.3), we find that

$$\boldsymbol{b}_2 = K\boldsymbol{a}_2 = KP_{21}\boldsymbol{b}_1 + KP_{22}\boldsymbol{b}_2,$$

whence $\boldsymbol{b}_2 = (E - P_{22}K)^{-1}KP_{21}\boldsymbol{b}_1$ and (3.5) follows. (3.5) is sometimes referred to as a **linear fractional transformation** and denoted by

$$LF(P;K).$$

Now in the general situation without assuming (3.3), we assume instead that $P_{21}$ is a (*square*) *regular* matrix (whence $q=r$). Then from the second equality of (3.6), we obtain

(3.7) $$\boldsymbol{b}_1 = P_{21}^{-1}(\boldsymbol{a}_2 - P_{22}\boldsymbol{b}_2).$$

Substituting (3.7) in the first equality of (3.6), we deduce that

(3.8) $$\boldsymbol{a}_1 = (P_{12} - P_{11}P_{21}^{-1}P_{22})\boldsymbol{b}_2 + P_{11}P_{21}^{-1}\boldsymbol{a}_2..$$

Hence putting

(3.9) $$CHAIN(P) = \begin{pmatrix} P_{12} - P_{11}P_{21}^{-1}P_{22} & P_{11}P_{21}^{-1} \\ -P_{21}^{-1}P_{22} & P_{21}^{-1}\end{pmatrix},$$



which is usually referred to as a **chain scattering representation** of $P$, we obtain an equivalent form of (3.1). Then we have

(3.10) $$\begin{pmatrix}\boldsymbol{a}_1\\ \boldsymbol{b}_1\end{pmatrix} = CHAIN(P)\begin{pmatrix}\boldsymbol{b}_2\\ \boldsymbol{a}_2\end{pmatrix}.$$

If we denote $CHAIN(P)$ by $G = \begin{pmatrix} G_{11} & G_{12}\\ G_{21} & G_{22}\end{pmatrix}$, then (3.10) reads

(3.11) $$\begin{pmatrix}\boldsymbol{a}_1\\ \boldsymbol{b}_1\end{pmatrix} = \begin{pmatrix} G_{11} & G_{12}\\ G_{21} & G_{22}\end{pmatrix}\begin{pmatrix}\boldsymbol{b}_2\\ \boldsymbol{a}_2\end{pmatrix}.$$

Substituting (3.3), (3.11) becomes

$$\begin{pmatrix}\boldsymbol{a}_1\\ \boldsymbol{b}_1\end{pmatrix} = \begin{pmatrix} G_{11}K + G_{12}\\ G_{21}K + G_{22}\end{pmatrix}\boldsymbol{a}_2,$$

whence we deduce that

(3.12) $$\begin{aligned}\Phi &= (G_{11}K + G_{12})(G_{21}K + G_{22})^{-1}\\ &= GK,\end{aligned}$$

the linear fractional transformation (which is referred to as a *homographic transformation* and denoted by $HM(\Phi;K)$ in [Kim]), where in the last equality we mean the action of $G$ on the variable $K$. We must impose the non-constant condition $|G| \neq 0$. Then $G \in GL_{m+r}(\mathbb{R})$ .

If $K$ is obtained from $L$ under the action of $H$, $K=HL$, then its composition with (3.12) yields $JL = \Phi = GHL$, i.e.

(3.13) $$J=GH,$$

which is referred to as the **cascade connection** or the cascade structure of $G$ and $H$.

Thus the chain-scattering representation of a system allows us to treat the feedback connection as a cascade connection.

Suppose a closed-loop system is given with $\boldsymbol{z} = \boldsymbol{a}_1 \in \mathbb{R}^m$, $\boldsymbol{y} = \boldsymbol{a}_2 \in \mathbb{R}^q$, $\boldsymbol{w} = \boldsymbol{b}_1 \in \mathbb{R}^r$ and $\boldsymbol{u} = \boldsymbol{b}_2 \in \mathbb{R}^p$ and $\Phi$ given by (3.2).

Then the $H^\infty$-**control problem** can be stated as follows. *Find a controller K such that the closed-loop system is internally stable and the transfer function* $\Phi$ *satisfies*

(3.14) $$\| \Phi \|_\infty < \gamma$$

*for a positive constant* $\gamma$ .

For the meaning of the norm, cf. §7.2.

## 4. Stability

Let $\boldsymbol{x} = \boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{u} = \boldsymbol{u}(t) \in \mathbb{R}^r$ and $\boldsymbol{y} = \boldsymbol{y}(t) \in \mathbb{R}^m$ be the **state** function, **input** function and **output** function, respectively. The system of DEs

(4.1) $$\begin{cases}\dfrac{d}{dt}\boldsymbol{x} &= A\boldsymbol{x} + B\boldsymbol{u}\\ \quad\boldsymbol{y} &= C\boldsymbol{x} + D\boldsymbol{u}\end{cases}$$

is called a **state equation** for a (time-invariant) **linear system**, where $A \in M_{n,n}(\mathbb{R})$, $B$, $C$ ,$D$ are given constant matrices. Using the matrix exponential function $e^{At}$– **state transfer matrix**, the first equation in (4.1) can be solved in the same way as for the scalar case:

(4.2) $$\begin{aligned}\boldsymbol{x} &= \boldsymbol{x}(t)\\ &= e^{At}\boldsymbol{x}(0) + Be^{At}\int_0^t e^{-A\tau}\boldsymbol{u}(t)\,\mathrm{d}\tau.\end{aligned}$$

**Definition 1**. *A linear system with the input* $u=o$

(4.3) $$\frac{d}{dt}\boldsymbol{x} = A\boldsymbol{x},$$

*called an* **autonomous system**, *is said to be* **asymptotically stable** *if for all initial values,* $\boldsymbol{x}(t)$ *approaches a limit as* $t\to\infty$.

Since the solution of (4.3) is given by

(4.4) $$\boldsymbol{x} = e^{At}\boldsymbol{x}(0),$$

the system is asymptotically stable if and only if $\| e^{At} \| \to 0$ as $t \to \infty$. Note that it is so if and only if the real parts of all eigenvalues of $A$ are negative.

A linear system is said to be **stable** if the step response of the system approaches a limit as times elapses, where **step response** means a response

(4.5) $$\boldsymbol{y}(t) = \int_0^t \boldsymbol{g}(t-\tau)\boldsymbol{u}_e(\tau)\,\mathrm{d}\tau = \int_0^t \boldsymbol{g}(t-\tau)\,\mathrm{d}\tau,$$

with the unit step function $\boldsymbol{u}_e = \boldsymbol{u}_e(t)$ as the input function and where $g(t)$ is the **impulse response**. In this case,

$$g(t) = Ce^{At}B + D\boldsymbol{\delta}(t)$$

## 5. Unity feedback system

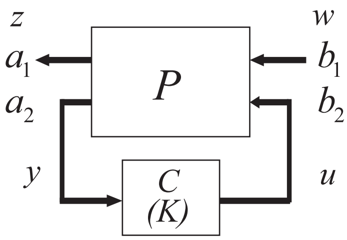


Figure 1.2

The synthesis problem of a controller of the unity feedback system depicted in Figure 1.2, refers to the **sensitivity reduction problem**, which asks for the estimation of the **sensitivity function** $S=S(s)$ multiplied by an appropriate frequency weighting function $W=W(s)$:

(5.1) $\qquad S=(E+PC)^{-1}$,

is a transfer function from $r$ to $e$, where $C=K$ is a (feed-forward) **compensator** and P is a plant. The problem consists in reducing the magnitude of $S$ over a specified frequency range $\Omega$, which amounts to finding a compensator $C$ stabilizing the closed-loop system such that

(5.2) $\qquad \| WS \|_\infty < \gamma$

for a positive constant $\gamma$.

To **accommodate this in the** $H^\infty$ **control problem**(3.14), we choose the matrix elements $P_{ij}$ of $P$ in such a way that the closed-loop transfer function $\Phi$ in (3.5) coincides with $WS$. First we are to choose $P_{22}=-P$. Then we would choose $P_{12}P_{21}=WP$. Then $\Phi$ becomes

$$P_{11}+WPC(E+PC)^{-1}$$
$$=P_{11}-W+W(E+PC)^{-1}.$$

Hence choosing $P_{11}=W$, we have $\Phi=WS$. Hence we may choose e.g.

(5.3) $$P=\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}=\begin{pmatrix} W & WP \\ E & -P \end{pmatrix}.$$

**Exercise 5.1** *Derive* (5.1) *directly from Figure 1.2.*

*Solution* Since $y=PC(r-y)$, it follows that $y=(E+PC)^{-1}PCr$. Since $PC$ is the open loop transfer function, we have $Sr$ is the tracking error for the input $r$.

## 6. Dualization

We may consider the dual chain-scattering representation of the plant $P$ in (3.2). We assume $P_{12}$ is a square invertible matrix (whence $m=p$). Then the argument goes in parallel to that leading to (3.10). Defining the **dual chain scattering matrix** by

(6.1) $$DCHAIN(P)=\begin{pmatrix} P_{12}^{-1} & P_{11}P_{12}^{-1} \\ -P_{12}^{-1}P_{22} & P_{21}-P_{22}P_{12}^{-1}P_{11} \end{pmatrix},$$

we obtain

(6.2) $\qquad CHAIN(P)\cdot DCHAIN(P)=E$

## 7. How to make critical reading

7.1. Analyticity of the Laplace transform

On [Kim, p.39] there is given a fallacious proof of the analyticity of the Laplace transform $F(s)$ of a function $f\in L_2^+$ which consists of those causal functions belonging to $L_2$ of square-integrable functions (in the sense of Lebesgue), i.e. those f's satisfying the conditions

$$f(t)=0 \quad (t<0); \quad ||f||_2=\sqrt{\int_0^\infty |f(t)|^2\,\mathrm{d}t}<\infty.$$

The statement is as follows with slight change of notation. Let

(7.1.1) $$F(s)=\int_0^\infty e^{-st}f(t)\,\mathrm{d}t$$

be the Laplace transform of $f$. Let $s=\sigma+j\omega$ and $\sigma>0$. Then, from (3.18), it follows that

(7.1.2) $$|F(s)|\le\int_0^\infty e^{-\sigma t}|f(t)|\,\mathrm{d}t<\infty.$$

This implies that $F(s)$ is analytic in the right half-plane $\sigma>0$.

Streamlining:

(7.1.2) should read

(7.1.3) $$\begin{aligned}&\int_0^\infty e^{-\sigma t}|f(t)|\,\mathrm{d}t\\ &\le\sqrt{\int_0^\infty e^{-2\sigma t}\,\mathrm{d}t}\sqrt{\int_0^\infty |f(t)|^2\,\mathrm{d}t}\\ &=\sqrt{\frac{1}{2\sigma}}||f||_2<\infty\end{aligned}$$

by the Cauchy-Schwarz inequality. Hence by the Weierstrass $M$-test, we conclude that the integral for $F(s)$ defines an analytic function in $\sigma>0$.

The Weierstrass $M$-test (Majorant series test) is most often used to assure the uniform convergence in case the series (integrals) are absolutely convergent, which in the case of integrals asserts that given a (complex-valued) function $f(x,y)$, $x\in[a,b]$ (resp$(a,b)$ as the case may be), $y\in Y$, if there is a positive (-valued) function $M(x)$ such that for any $y\in Y$,

$$|f(x,y)|\le M(x)$$

and

$$\int_a^\infty M(x)\,\mathrm{d}x<\infty \quad (\text{resp.} \int_a^b M(x)\,\mathrm{d}x<\infty),$$

then $\int_a^\infty f(x,y)\,\mathrm{d}x$ (resp. $\int_a^b M(x)\,\mathrm{d}x<\infty$) is absolutely and uniformly convergent on $Y$.

Then we apply another theorem which is a variant of the Wierstrass double series theorem to the effect that a uniformly convergent integral (series) defines an analytic function in the domain of uniform convergence.
Following Titchmarsh [Tit], we often refer to this as "by absolute convergence".

7.2. Norm of the Euclidean spaces

The norm of a matrix seems to be first given on [Kim, p. 31] without any definition given, while the norm of a vector is given in [Kim, (3.10), p. 38]: The norm

$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^n$ is defined to be the Euclidean norm

(7.2.1) $$||\boldsymbol{x}|| = ||\boldsymbol{x}||_2 = \sqrt{\sum_{j=1}^{n} |x_j|^2}$$

or by the sup norm

(7.2.2) $$||\boldsymbol{x}|| = ||\boldsymbol{x}||_\infty = \max\{|x_1|, \cdots. |x_n|\},$$

or anything that satisfies the axioms of the norm. They define the same topology on $\mathbb{C}^n$.

The definition of the norm of a matrix should be given in a similar way by viewing its elements as an $n^2$-dimensional vector, i.e. embedding it in $\mathbb{C}^{n^2}$. If $A=(a_{ij})$, $1 \le i, j \le n$, then

(7.2.3) $$||A|| = ||A||_2 = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}$$

or otherwise.

We shall prove the following whose proof can be readily generalized to give

(7.2.4) $$\lim_{p\to\infty} ||f||_p = ||f||_\infty = \sup_{t\ge 0} |f(t)|.$$

For $\boldsymbol{a}=(a_1,\cdots,a_n)$,

(7.2.5) $$\lim_{p\to\infty} ||\boldsymbol{a}||_p = \lim_{p\to\infty} \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} = ||\boldsymbol{a}||_\infty = \max_{1\le k\le n}\{|a_k|^p\}.$$

For suppose $|a_1| = \max_{1\le k\le n}\{|a_k|^p\}$. Then for any $>0$,
$|a_1| = (|a_1|^p)^{\frac{1}{p}} \le (\sum_{k=1}^{n} |a_k|^p)^{\frac{1}{p}}$
On the other hand, since $|a_1| \ge |a_k|, 1 \le k \le n$, we obtain

(7.2.6) $$\left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} = |a_1|(1 + \sum_{k=2}^{n} \left|\frac{a_k}{a_1}\right|^p)^{\frac{1}{p}} \le |a_1|(1+n-1)^{\frac{1}{p}}$$

For $p>1$, the Bernoulli inequality gives $(1+n-1)^{\frac{1}{p}} \le 1+\frac{n-1}{p} \to 1$ as $p\to \infty$. Hence the right-hand side of (7.2.6) tends to $|a_1|$.
The $p$-norm in (7.2.4) is defined by

$$||f||_p = \left(\int_0^\infty ||f(t)||^p \,\mathrm{d}t\right)^{\frac{1}{p}},$$

where $\| f(t) \|$ is any Euclidean norm. However, the restriction that $\| f(t) \| \to 0$ as $t\to\infty$ excludes signals of infinite duration such as unit step signals or periodic ones from $L_p$. To circumvent the inconvenience, the **power** is introduced:

(7.2.7) $$\mathrm{power}(f) = \lim_{T\to\infty} \left(\frac{1}{T}\int_0^T ||f(t)||^2 \,\mathrm{d}t\right)^{\frac{1}{2}}.$$

However, it is not checked whether the limit in (7.2.7) exits.

On [Kim, p. 39], it is stated that the 2-norm $\| \cdot \|_2$ is important because it is induced from the inner product

(3.17) $$< f, g \ge \int_0^\infty f^*(t)g(t)\,\mathrm{d}t, \quad ||f||_2 = \sqrt{< f, f >},$$

where* refers to the transposed complex conjugation (seems to be first given on p. 32). However, the most important property of these function spaces is not mentioned, i.e. that $L^p$ is a **Banach space** (i.e. a **complete** metric space) and in particular $L^2$ is a **Hilbert space**. Even we cannot perceive which integral is meant or that the functions are not ordinary functions but classes of functions which are regarded as the same if they differ only at measure 0 set. The author refers to the **Parseval identity**[Kim, (3.19)] for which it is essential that the system be complete.

On [Kim, p. 39]the **Hardy space**$H_p$ is introduced which is the space of all f(s) which are analytic in $\mathcal{RHP}$ —RHP– right half-plane $\sigma>0$ such that $f(j\omega)\in L_p$; in particular, $H_\infty$ with sup norm. Thus $H^\infty$-control problem is about those (rational) functions which are analytic $\mathcal{RHP}$, *a fortiori* stable, with regard to the sup norm.

7.3. Integral transforms

We state the Mellin, (two-sided) Laplace and the Fourier transforms. If $f(x)=O(x^\alpha)$, $x>0$, then its **Mellin transform** $M[f]$ is defined by

(7.3.1) $$M[f](s) = \int_0^\infty x^s f(x) \frac{\mathrm{d}x}{x}, \quad \sigma > \alpha.$$

Under the change of variable $x=e^{-t}$, the Mellin transform and the **two-sided Laplace transform** shift each other:

(7.3.2) $$L^{\pm}[\varphi](s) = \int_{-\infty}^\infty e^{-st}\varphi(t)\,\mathrm{d}t, \quad \sigma > \alpha,$$

where we write $\varphi(t) = f(e^{-t})$.

The ordinary **Laplace transform** (one-sided Laplace transform) is obtained by multiplying the integrand by the unit step function $u_e = u_e\ (t)$ in §4.5, which is 0 for $t<0$ and 1 for $t \geq 0$:

$$L[f](s) = L^{\pm}[fu](s) = \int_0^{\infty} e^{-st} f(e^{-t})\, \mathrm{d}t, \quad \sigma > \alpha, \tag{7.3.3}$$

If we fix $\kappa > \alpha$ and write $s = \kappa + i\omega$, $G(\omega) = L^{\pm}[f](\varkappa + i\omega)$ $g(t) = e^{-\varkappa t} f(e^{-t})$ in (7.3.2), then it changes into

$$G(\omega) = F[g](\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} g(t)\, \mathrm{d}t = L^{\pm}[\varphi](i\omega), \tag{7.3.4}$$

the **Fourier transform** of $g$. The Fourier transform is the linking key of linear control theory and coding theory to be presented in the fourth seminar.

We explain Plancherel's theorem for functions in $L_2(\mathbb{R})$ . Let

$$\hat{f}_T(x) = \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} e^{-ixt} f(t)\, \mathrm{d}t. \tag{7.3.5}$$

Then $\hat{f}_T(x)$ converges to a function $\hat{f}$ in $L^2$:

$$||\hat{f}_T - \hat{f}|| \to 0 \quad T \to \infty; \quad \mathrm{l.i.m.}_{T\to\infty} \hat{f}_T(x) = \hat{f}(t), \tag{7.3.6}$$

wherel.i.m. is a short-hand for "limit in the mean". The **Parseval identity** reads

$$||\hat{f}||_2 = ||f||_2, \quad \int_{-\infty}^{\infty} |\hat{f}(t)|^2\, \mathrm{d}t = \int_{-\infty}^{\infty} |f(t)|^2\, \mathrm{d}t. \tag{7.3.7}$$

If we apply (7.3.7) to a causal function $f$, then it leads to [Kim,(3.19)]

$$\int_{-\infty}^{\infty} |L^{\pm}[f](i\omega)|^2\, \mathrm{d}\omega = \int_0^{\infty} |f(t)|^2\, \mathrm{d}t. \tag{7.3.8}$$

Hence we see that [Kim,(3.19)] is indeed the Parseval identity for the Fourier (or Plancherel) transform for $f \in L_2(\mathbb{R})$.

In the discussion that followed the seminar, we made clear some close relations among those topics about which we are concerned. I.e. linear algebra can be the centripetal force which attracts, as its second course, control of linear systems, man-machine control in human factors and coding theory.

## 8. Intermezzo

We take the policy of choosing a few topics from [Nag] and explain them in a lucid way, by making them accessible to everyone. The topics we choose include

**(i) Man-machine system**[Nag, pp. 90-94],
**(ii) Discrete Fourier analysis**[Nag, pp. 76-81],
**(iii) Noise pollution and its prevention**[Nag, pp. 121-126]
**(iv) Internal clocks and labor**[Nag, pp. 21-22, 177],

whereby we enrich the contents by adding some more material from references [Hay] to (i), [ATW], [CT], [Wea] to (ii), and [Kucz], [Luc], [Mor], [Nak] to (iii), respectively.

## 9. [Nag, pp. 76-81] Discrete Fourier transforms

It is often said, especially by audience which has trained musical ears, that music played by digital devices sound rather **flat** and even boring compared with live performance. There is a good reason for that. In transforming analogue signal into digital signal, the A/D transformer being used, which makes quantization and approximates the sampled signal at digital levels. This **rounding-off** of analogue signals give them a sort of unpleasant perfection. A good example is a karaoke estimator which would give bad marks for those professional singers who can get the audience moved, on the ground that they don't sing as in the scores.

— "As is true with all artifacts, completed ones are less interesting; those which are unfinished and left at that are more appealing and give liveliness." from *Anhermite's miscellany* by K. Yoshida.

—Hans describes the crystal of snow as "This is too regular. A living thing cannot be so symmetric. Things with such perfection smell of death. The Creator, it seems, has designed all that exists in his world in a fashion slightly deviated from perfect symmetry"—*Der Zauberberg* by Thomas Mann.

Thus digital art smells of death.

**Exercise 9.1.** *For your fun, try to find (at least) two linguistic jokes contained in this note.*

In electrical engineering, with $\omega$ being the frequency, the variable is often denoted by $s = \sigma + \mathrm{j}\omega$ ($j$ being the imaginary unit) or $p = \sigma + j\omega$. The celebrated sampling theorem ([Wea, pp. 116-120]) is essential in frequency analysis:

**Theorem 9.1.** If the function *f(t)is band-limited with band-length* $2\Omega$:

$$\hat{f}(\omega) = 0 \quad |\omega| \geq \Omega > 0, \tag{9.1}$$

*then (discretized) f(t) is uniquely determined by knowledge of its (discretized) values at uniformly-spaced intervals* $\Delta t = \frac{1}{2\Omega}$ *apart*:

(9.2) $$f(t)=\sum_{n=-\infty}^{\infty} f\,(n\Delta t)\frac{sin\ (\,2\pi\Omega(t-n\Delta t))}{2\pi\Omega(t-n\Delta t)}.$$

The sampling rate $\Delta t=\frac{1}{2\Omega}$ is often called the *Nyquist rate* and the sequence obtained with this rate is called the *Nyquist sampling.*

We use the subsequent argument from[Vista I, pp. 154-156] whose slight modification gives a proof of the above theorem.

Use being made of the Fourier transform $\hat{f}(\omega)$, the original function $f(t)$ which behaves differently on different parts of the t-domain ($t$-axis) may be expressed by a unique formula

(9.3) $$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\,e^{i\omega t}\mathrm{d}\omega.$$

This is called the **Fourier Integral Theorem**. To be more precise, if $f, f'$ are piecewise continuous and

$$\int_{-\infty}^{\infty}|f(t)|\mathrm{d}t<\infty,$$

then the theorem holds in the following form:

(9.4) $$\frac{1}{2}\{f(t+0)+f(t-0)\} =\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\,e^{it\omega}\mathrm{d}\omega.$$



If we define the **inverse Fourier transform**$F^{-1}f$ of $f$ by

(9.5) $$(\mathcal{F}^{-1}f)(\omega)=\int_{-\infty}^{\infty}f(t)\,e^{it\omega}\mathrm{d}t,$$

then we may express the Fourier Integral Theorem as $F^{-1}Ff=f$ or $\hat{\hat{f}}(-t)=f(t)$. (9.3) may be deduced formally from [Vista I, Theorem 7.2,p. 141] as follows.

If $f$ is piecewise smooth and continuous on $[-T,T]$, then it can be expanded into a Fourier series:

(9.6) $$f(t)=\sum_{n=-\infty}^{\infty}c_n\,e^{i\lambda_n t},$$
$$c_n=\frac{1}{2T}\int_{-T}^{T}f(t)\,e^{-i\lambda_n t}\mathrm{d}t,$$
$$\lambda_n=\frac{2\pi}{2T}n,\quad |t|<T.$$

Letting $T\to\infty$, $n\to\infty$, we may contend that

(9.7) $$2Tc_n\to\int_{-\infty}^{\infty}f(t)\,e^{-i\lambda_n t}\mathrm{d}t=\hat{f}(\lambda_n)$$

and therefore

(9.8) $$f(t)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}2\,Tc_n e^{i\lambda_n t}(\lambda_{n+1}-\lambda_n)$$

(9.9) $$\to\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\hat{f}\,(\lambda_n)e^{i\lambda_n t}\Delta\lambda_n \sim\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\,e^{it\omega}\mathrm{d}\omega.$$

Viewing this as

$$f(t)=\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)\,e^{-it\omega}\mathrm{d}t\right)e^{i\omega t}\mathrm{d}\omega,$$

this may be thought of as giving the motivation for the definition of $\hat{f}(\omega)$.

Also using the defining equation

$$\int_{-\infty}^{\infty}f(x)\,\delta(t-x)\mathrm{d}x=f(t)$$

for the delta function and one of its well known properties

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ixt}\mathrm{d}x=\delta(t),$$

we can give the following simple proof.

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}(Ff)(y)\,e^{iyt}\mathrm{d}y\\ &=\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ixy}f(x)\mathrm{d}x\right)e^{iyt}\mathrm{d}y\\ &=\int_{-\infty}^{\infty}f(x)\mathrm{d}x\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{iy(t-x)}\mathrm{d}y\right)\\ &=\int_{-\infty}^{\infty}f(x)\,\delta(t-x)\mathrm{d}x=f(t).\end{aligned}$$

This is a legitimate proof if the inversion of the order of integration is justified. Another proof can be given in the spirit of Proof of [Vista I, Theorem 7.2], which will be given elsewhere. In view of the appearance of the factor $\frac{1}{2\pi}$ in the Fourier integral theorem, we often introduce normalization by distributing it to both transforms. The symmetric **pair of the Fourier transform** and the inverse Fourier transform is

(9.10) $$(Ff)(y)=\hat{f}(y) =\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ixy}f(x)\mathrm{d}x,$$

(9.11) $$(F^{-1}f)(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,e^{ixy}\mathrm{d}x.$$

In view of (9.10), we may also consider the case where there does not appear the preceding factor:

(9.12) $$(\mathcal{F}f)(\omega)=\hat{f}(\omega) =\int_{-\infty}^{\infty}e^{-2\pi i\omega t}f(t)\mathrm{d}t,$$

(9.13) $$(\mathcal{F}^{-1}f)(t)=\int_{-\infty}^{\infty}f(\omega)\,e^{2\pi it\omega}\mathrm{d}\omega.$$

Then the Fourier cosine transform takes the form for f even,

$$(Cf)(s)=\sqrt{\tfrac{1}{2\pi}}\int_{-\infty}^{\infty}\cos(xs)\,f(x)\mathrm{d}x =\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\cos(xs)\,f(x)\mathrm{d}x$$

and if f is odd, then the Fourier sine transform becomes

$$(Sf)(s)=\sqrt{\tfrac{1}{2\pi}}\int_{-\infty}^{\infty}\sin(xs)\,f(x)\mathrm{d}x =\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\sin(xs)\,f(x)\mathrm{d}x.$$

Now we may give a proof of the sampling theorem. Correspondingly to (9.6), we have

(9.14)
$$\hat{f}(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{2\pi i\lambda_n \omega}$$
$$c_n = \frac{1}{2\Omega}\int_{-\Omega}^{\Omega} \hat{f}(\omega)\, e^{-2\pi i\lambda_n\omega} \mathrm{d}\omega,$$
$$\lambda_n = \frac{1}{2\Omega} n,$$

where $\hat{f}$ is the periodic extension of $\hat{f}$. Then differently from (9.7), we have the equality

(9.15)
$$2\Omega c_n = \int_{-\Omega}^{\Omega} \hat{f}(\omega)\, e^{-2\pi i\lambda_n\omega} \mathrm{d}\omega$$
$$= \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-2\pi i\lambda_n t} \mathrm{d}\omega = f(-\lambda_n)$$

Then correspondingly to (9.8), we deduce that

(9.16)
$$\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} 2\,\Omega c_n e^{2\pi i\lambda_n t}\frac{1}{2\Omega}$$
$$= \sum_{n=-\infty}^{\infty} f(-\lambda_n) e^{2\pi i\lambda_n t}\frac{1}{2\Omega}.$$

Applying the Fourier inversion (9.12) bearing in mind (9.1), we obtain

(9.17)
$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{2\pi i t\omega} \mathrm{d}\omega$$
$$= \int_{-\Omega}^{\Omega} \hat{f}(\omega)\, e^{2\pi i t\omega} \mathrm{d}\omega.$$

Substituting (9.16) into (9.17) and assuming that the *interchange of integration and summation is legitimate*, we obtain

(9.18)
$$f(t) = \sum_{n=-\infty}^{\infty} f(-\lambda_n)\frac{1}{2\Omega}\int_{-\Omega}^{\Omega} e^{2\pi i(t+\lambda_n)\omega} \mathrm{d}\omega.$$

Let $\{f(k)\,|\,0 \le k \le N-1\}$ be a sequence of $N$ terms (or a periodic sequence of period $N$). Then the **discrete Fourier transform** $\hat{f}$ of $f$ is defined e.g. on [Wea, p. 91] by

(9.19)
$$\hat{f}(j) = \frac{1}{N}\sum_{k=0}^{N-1} f(k) e^{-2\pi i\frac{j}{N}k}, 0 \le j \le N-1$$

and it can be inverted as

(9.20)
$$f(k) = \sum_{j=0}^{N-1} \hat{f}(j) e^{2\pi i\frac{k}{N}j}, 0 \le k \le N-1.$$

Recall the following fundamental theorem in algebra (referred to in the fourth seminar).

**Theorem 9.2**. *Let* $\Omega$ *be an algebraically closed field with characteristic* $p$ *(prime or 0). The following statements are equivalent.*

(1)*The set of all m-th roots of* 1 *forms a cyclic group of order m, whence there exists a primitive m-th root of* 1.

(2)*p does not divide m.*

Let $\zeta = \zeta_N$ be a primitive $N$-th root of unity which exists on the ground of Theorem 2 ($\zeta = e^{2\pi i\frac{1}{N}}$ in the case of C) called the weighting kernel in [Wea]. With it, Eqs.(9.19) and(9.20) can be expressed as

(9.21)
$$\hat{f}(j) = \frac{1}{N}\sum_{k=0}^{N-1} f(k)\zeta^{-jk}, 0 \le j \le N-1$$

and its inverse relation

(9.22)
$$f(k) = \sum_{j=0}^{N-1} \hat{f}(j)\zeta^{kj}, \quad 0 \le k \le N-1.$$

"The discrete Fourier transform links the human factors to coding theory."

Let $F$=GF($q$) be the finite field with $q=p^m$ elements, with $p$ a prime and m a positive integer. Let $N$ be relatively prime to p. Then there exists a primitive $N$-th root $\zeta$ in an extension field $E$ of $F$. If we write the sequence as $\boldsymbol{a}=(a_0, \cdots, a_{N-1}) \in F^n$. Then the discrete Fourier transform DFT($\boldsymbol{a}$) is given by $\hat{\boldsymbol{a}}$ =DFT($\boldsymbol{a}$)=$(\hat{a}_0, \cdots, \hat{a}_{N-1}) \in E^N$, where $\hat{a}_j$ is defined by (9.21) (sometimes without the factor $1/N$). The **circulant matrix** formed from DFT($\boldsymbol{a}$) is called the DFT matrix $M(\boldsymbol{a})$ of $\boldsymbol{a}$:

(9.23)
$$M(\boldsymbol{a}) = \begin{pmatrix} \hat{a}_0 & \hat{a}_1 & \cdots & \hat{a} \\ \hat{a}_1 & \hat{a}_2 & \cdots & \\ & \cdots & & \\ \hat{a}_{N-1} & \hat{a}_0 & \cdots & \hat{a} \end{pmatrix}$$

The Blahut theorem asserts that "The Hamming weight (the number of non-zero terms) $w(\boldsymbol{a})$ is the rank of the DFT matrix $M(\boldsymbol{a})$."

This has been continued in the fourth seminar. We suspect that we could apply the rich theory of circulant matrices to deduce non-trivial results on the above.

## 10. Internal clocks and labor

All living creatures have two internal rhythms, one is based on the rhythm of the sun and the other on that of the moon. The former is called the **circadian rhythm** and consists of 24 hours while the latter is based on the rhythm of the tide and consists of about 25 hours, about 50 minutes longer than the circadian rhythm, which we refer to as the **tidal rhythm**. Thus there are two "internal clocks" (body clocks) in living organisms.

It seems that the sleep, which is usually to be taken during night, is governed by the tidal rhythm. Then as is clear from the context of the tidal rhythm, every day there occurs a delay of about 50 minutes before going to sleep. Therefore, for those who cannot regulate their sleeping cycles to this delay may start suffering **insomnia.** The sleep consists of two types of sleep cycles− NON-REM (ortho) cycle and REM (Rapid Eye Movement (para) cycle, in which one is dreaming. The former is the sleep of the brain and the latter is that of the body [Nag, pp. 21-22]. A good sleep must include a few times of these cycles and should end with the REM.

According to [Weil], all kinds of oppressive tranquilizers give bad effects on the REM sleep and get rid of dreams which are indispensable for the health. Weil recommends to take the roots of "ValerianaOfficialis" in tincture which is available at a food supplement shops, where recently, more effective medicine is available, the synthesized "melatonin", which is a hormone (generated in brain) regulating the rhythm of day and night, and is thought of as adjusting the body clock. There are many examples of people who had been suffering from the jet lag got cured after taking the melatonin. Jet lag insomnia occurs because of the inversion of day and night and is often common with those businessmen who travel from the West to the East (because of the counter-clockwise circulation of the Earth). They catch up with and pass by the day time next day back in their own places. However, Kawamura [Kaw] is denying the effect of melatonin and refers to the circadian rhythm as 24±5 hours.



In summary, the cycle of reverse implications look rather persuasive:

- melatonin cures time lag insomnia → melatonin fixes the inversion of day and night → insomnia may occur from the inversion of day and night → the inversion of day and night occurs because of the shift of the tidal rhythm.

There are many other outer causes which disturb good sleep. The biggest one may be the noise problem expounded in §11. It is said that at 50 phon (dB), sleep (and emotion) is disturbed. Continual noise gives more trouble than continuous one. E.g. the dog barking may give a big trouble because of its musical property. In such cases, a white noise generator or a noise canceller can be used. The former generates the white noise similar to the sound of the flow of water and the latter makes an immediate frequency analysis of noise and generates its mirror reflection to cancel it. Now noise-cancellers are available from some special shops.

Now we turn to the circadian (or rather) rhythm and the work efficiency as measured by CFF (**critical fusion frequency** of flicker) test. Cf. [Nag, p. 53-55, p. 177]. We note that the CFF test apparently shows the tidal rhythm of the brain as the 50 min. delay under the conditions that the subject cannot distinguish day and night nor notice the time and is asked to keep performing a given work. The conditions are the same as those under which the tidal rhythm has been measured. It follows that in the night shift, there can occur more often human error, which should be put into consideration for designing the plan for work.

Under the modern working conditions, workers cannot distinguish day and night, nor time, and are feasible to tidal rhythm, which should be considered in assigning night shift work.

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