The singular interactions between a periodic soliton and an infinitesimal periodic soliton: Solutions to the Davey-Stewartson II equation

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Abstract

There is a very small but finite amplitude periodic soliton (an infinitesimal periodic soliton) in the DS II equation with $r > 0$ which interacts resonantly with a finite-amplitude periodic soliton under certain conditions. It is shown that these interactions always become parameter-sensitive.

1 Introduction

The Davey-Stewartson (DS) equation is expressed as [1]

$$\left\{ \begin{array}{l} iu_t + pu_{xx} + u_{yy} + r|u|^2 u - 2uv = 0, \\ v_{xx} - pv_{yy} - r(|u|^2)_{xx} = 0, \end{array} \right. \tag{1}$$

where $p = \pm 1$ and r is a constant. Equation (1) with $p = 1$ and $p = -1$ is called the DS I and DS II equations, respectively. The solutions to the DS equation were obtained previously in various aspects. The N-soliton solutions of the DS equation were obtained by Anker and Freeman by the inverse-scattering method [2], and later by Satsuma and Ablowitz by the Hirota method [3].

The DS I equation has line soliton solution for which the wave numbers and frequency consist of only real number components and periodic soliton solution for which they consist of real number components and imaginary components. It is well known that resonant interaction exists between these solitons [4, 5]. Further, it has been shown recently that in parameter space of periodic soliton solution, the solution having parameters on the boundary surface where the solution changes from regular to nonregular is regarded as line soliton solution [6]. This soliton that has parameters in the vicinity of the boundary surface is called quasi-line soliton. The quasi-line soliton, unlike a line soliton, has imaginary components of the wave numbers and frequency as hidden parameters. Further, shape of the quasi-line soliton is almost same as that of the line soliton, being slightly different only in its spreading form. Recent studies have revealed that an interesting singular interaction that cannot be seen in the line soliton exists in the soliton resonance that quasi-line soliton is related to, and that a parameter-sensitive phenomenon in which a state of the interaction varies drastically by slight variation of parameters exists in this interaction [6].

On the other hand, the DS II equation with $r > 0$ has the periodic soliton solution but not the line soliton solution. It is known that this periodic soliton solution is regular in all regions of parameter space and that resonant interaction exists between the periodic solitons [7]. The purpose of this study is to investigate to find if parameter-sensitive singular interaction exists similar to the DS I equation in the periodic soliton resonance in the DS II equation with $r > 0$.

This paper is summarized as follows. In Sec. II, we introduce the infinitesimal periodic soliton, which has extremely small amplitude. In Sec III, we examine the interaction between a periodic soliton and an infinitesimal-periodic soliton. It is shown there are singular interactions between a periodic soliton and an infinitesimal-periodic soliton under certain conditions and that there are parameter-sensitive regions in the parameter space of the two-periodic-soliton solution. Conclusions are presented in Sec. IV. The contents of this paper are a summarized version of the paper that has already been published [8]. Moreover, this study is a collaborative research performed with Prof. Masayoshi Tajiri (Osaka Prefecture University).

2 Infinitesimal periodic soliton

The periodic soliton solution to the DS II equation with $r > 0$ is given by

$$u = u_0 e^{i(\zeta+\phi_r)} \frac{\cosh(\xi + i\phi_r) + \frac{1}{\sqrt{M}}\cos(\eta + i\phi_i)}{\cosh\xi + \frac{1}{\sqrt{M}}\cos\eta}, \tag{2}$$

$$v = 2\frac{\alpha^2 - \frac{\beta^2}{M} + \frac{\alpha^2-\beta^2}{\sqrt{M}}\cosh\xi\cos\eta + \frac{2\alpha\beta}{\sqrt{M}}\sinh\xi\sin\eta}{\left(\cosh\xi + \frac{1}{\sqrt{M}}\cos\eta\right)^2}, \tag{3}$$

where

$$\begin{aligned}
&\zeta = kx + ly - \omega t + \zeta_0,\\
&\xi = \alpha x + \gamma y - \Omega_r t + \xi^0,\\
&\eta = \beta x + \delta y - \Omega_i t + \eta^0,\\
&\omega = -k^2 + l^2 - ru_0^2,\\
&\sin^2\frac{\phi}{2} = -\frac{(\alpha + i\beta)^2 + (\gamma + i\delta)^2}{2ru_0^2},
\end{aligned} \tag{4}$$

$$\Omega_r + i\Omega_i = -2k(\alpha + i\beta) + 2l(\gamma + i\delta) + \left\{(\alpha + i\beta)^2 - (\gamma + i\delta)^2\right\}\cot\frac{\phi}{2}, \tag{5}$$

$$M = \frac{2ru_0^2\sin\frac{\phi}{2}\sin\frac{\phi^*}{2}\cos\frac{\phi-\phi^*}{2} + \left\{(\alpha+i\beta)(\alpha-i\beta) + (\gamma+i\delta)(\gamma-i\delta)\right\}}{2ru_0^2\sin\frac{\phi}{2}\sin\frac{\phi^*}{2}\cos\frac{\phi+\phi^*}{2}\left\{(\alpha+i\beta)(\alpha-i\beta) + (\gamma+i\delta)(\gamma-i\delta)\right\}}, \tag{6}$$

where $\phi = \phi_r + i\phi_i$. If we express the complex wave number $(\alpha + i\beta, \gamma + i\delta)$ in terms of ϕ and θ as

$$\begin{cases} \alpha + i\beta = i\sqrt{2ru_0^2}\sin\frac{\phi}{2}\cos\theta, \\ \gamma + i\delta = i\sqrt{2ru_0^2}\sin\frac{\phi}{2}\sin\theta, \end{cases} \tag{7}$$

then equaiton (6) is expressed as

$$M = \frac{\cosh\phi_i + \cosh 2\theta_i}{\cos\phi_r + \cosh 2\theta_i}, \tag{8}$$

where $\theta = \theta_r + i\theta_i$. Equation (8) presents $M > 1$ for all parameters. Therefore, no conditions were set for a singular solution, and namely, the solution is regular for all the parameters. The periodic soliton solution of the DS I equation becomes a line soliton soliton by taking the limit $M \to \infty$. Here, we examine the structure of the periodic soliton solution of the DS II equation when the limit of $M \to \infty$ of the periodic soliton solution is applied. Equation (8) shows that the value of M becomes infinite reaching the limit.

$$\cos\phi_r + \cosh 2\theta_i \to +0. \tag{9}$$

From the condition $\cos\phi_r = -\cosh 2\theta_i$, the following is obtained, with n as an integer,

$$\phi_r = (2n+1)\pi, \qquad \theta_i = 0. \tag{10}$$

Subsituting equation (10) into equations (5) and (7), we see

$$\begin{cases} \alpha = 0, \\ \gamma = 0, \\ \beta = \beta_0 = \sqrt{2ru_0^2}\cosh\frac{\phi_i}{2}\cos\theta_r, \\ \delta = \delta_0 = \sqrt{2ru_0^2}\cosh\frac{\phi_i}{2}\sin\theta_r, \end{cases} \tag{11}$$

and

$$\begin{cases} \Omega_r = 0, \\ \Omega_i = \Omega_{i0} = -2k\beta_0 + 2l\delta_0 + (\beta_0^2 - \delta_0^2)\tanh\frac{\phi_i}{2}. \end{cases} \tag{12}$$

Equations (2) and (3) become

$$u = u_0 e^{i\zeta}, \qquad v = 0. \tag{13}$$

Therefore, although imaginary wave numbers and frequency are finite values (not zero), no solitary wave is obtained at the limit $M \to \infty$. The periodic soliton disappears at the parameter point where M becomes infinite. However, in the case of taking the parameters ϕ_r and θ_i as

$$\phi_r = (2n+1)\pi + 2\bar{\varepsilon}_1, \qquad \theta_i = \bar{\varepsilon}_2, \tag{14}$$

M is given by

$$M = \frac{1+\cosh\phi_i}{2(\bar{\varepsilon}_1^2 + \bar{\varepsilon}_2^2)} \sim O(1/\bar{\varepsilon}^2), \tag{15}$$

and

$$\begin{cases} \alpha = \sqrt{2ru_0^2}\left\{\bar{\varepsilon}_1\sinh\frac{\phi_i}{2}\cos\theta_r + \bar{\varepsilon}_2\cosh\frac{\phi_i}{2}\sin\theta_r + O(\bar{\varepsilon}^3)\right\} \sim O(\bar{\varepsilon}), \\ \gamma = \sqrt{2ru_0^2}\left\{\bar{\varepsilon}_1\sinh\frac{\phi_i}{2}\sin\theta_r - \bar{\varepsilon}_2\cosh\frac{\phi_i}{2}\cos\theta_r + O(\bar{\varepsilon}^3)\right\} \sim O(\bar{\varepsilon}), \\ \beta = \beta_0 - \sqrt{2ru_0^2}\left\{\bar{\varepsilon}_1\bar{\varepsilon}_2\sinh\frac{\phi_i}{2}\sin\theta_r + \frac{\bar{\varepsilon}_1^2-\bar{\varepsilon}_2^2}{2}\cosh\frac{\phi_i}{2}\cos\theta_r + O(\bar{\varepsilon}^3)\right\} \sim O(1), \\ \delta = \delta_0 + \sqrt{2ru_0^2}\left\{\bar{\varepsilon}_1\bar{\varepsilon}_2\sinh\frac{\phi_i}{2}\cos\theta_r - \frac{\bar{\varepsilon}_1^2-\bar{\varepsilon}_2^2}{2}\cosh\frac{\phi_i}{2}\sin\theta_r + O(\bar{\varepsilon}^3)\right\} \sim O(1), \end{cases} \tag{16}$$

and

$$\begin{cases} \Omega_r = -2k\alpha + 2l\gamma + 2(\alpha\beta_0 - \gamma\delta_0)\tanh\frac{\phi_i}{2} + \bar{\varepsilon}_1(\beta_0^2 - \delta_0^2)\mathrm{sech}^2\frac{\phi_i}{2} + O(\bar{\varepsilon}^3), \\ \Omega_i = \Omega_{i0} + O(\bar{\varepsilon}^2), \end{cases} \tag{17}$$

and

$$u = u_0 e^{i(\zeta + 2\bar{\varepsilon}_1)}\left\{1 - \frac{2\cosh^2\frac{\phi_i}{2}}{\sqrt{M}}\mathrm{sech}\xi\cos\eta + i\left(2\bar{\varepsilon}_1\tanh\xi + \frac{\sinh\phi_i}{\sqrt{M}}\mathrm{sech}\xi\sin\eta\right) + O(\bar{\varepsilon}^2)\right\}, \tag{18}$$

$$v = -2\frac{\beta^2}{\sqrt{M}}\mathrm{sech}\xi\cos\eta + O(\bar{\varepsilon}^2). \tag{19}$$

It is shown that the amplitude of the periodic soliton decreases with an increase of the value of M. This extremely small amplitude periodic soliton characterized by $\phi_r = (2n+1)\pi + 2\bar{\varepsilon}_1$, $\theta_i = \bar{\varepsilon}_2$ is to be called an infinitesimal periodic soliton in this paper.

3 Interactions between periodic soliton and infinitesimal periodic soliton

It is well known that the solution describing the interaction between two periodic solitons is given as

$$u = \frac{g}{f}, \qquad v = 2(\ln f)_{xx}, \tag{20}$$

with

$$f = 1 + \frac{M_1}{4}e^{2\xi_1} + \frac{M_2}{4}e^{2\xi_2} + \frac{M_1 M_2 L_1^2 L_2^2}{16}e^{2(\xi_1+\xi_2)} + e^{\xi_1}\left\{\cos\eta_1 + \frac{M_2 L_1 L_2}{4}e^{2\xi_2}\cos(\eta_1+\varphi_1+\varphi_2)\right\}$$
$$+e^{\xi_2}\left\{\cos\eta_2 + \frac{M_1 L_1 L_2}{4}e^{2\xi_1}\cos(\eta_2+\varphi_1-\varphi_2)\right\}$$
$$+\frac{1}{2}e^{\xi_1+\xi_2}\left\{L_1\cos(\eta_1+\eta_2+\varphi_1) + L_2\cos(\eta_1-\eta_2+\varphi_2)\right\}, \tag{21}$$

$$g = u_0 e^{i\zeta} f(\xi_1 + i\phi_{1r},\ \xi_2 + i\phi_{2r},\ \eta_1 + i\phi_{1i},\ \eta_2 + i\phi_{2i}), \tag{22}$$

where

$$\xi_j = \alpha_j x + \gamma_j y - \Omega_{jr} t + \xi_j^0,$$
$$\eta_j = \beta_j x + \delta_j y - \Omega_{ji} t + \eta_j^0,$$
$$\sin^2\frac{\phi_{jr}+i\phi_{ji}}{2} = -\frac{(\alpha_j+i\beta_j)^2 + (\gamma_j+i\delta_j)^2}{2ru_0^2}, \tag{23}$$
$$\Omega_{jr} + i\Omega_{ji} = -2k(\alpha_j+i\beta_j) + 2l(\gamma_j+i\delta_j)$$
$$+\left\{(\alpha_j+i\beta_j)^2 - (\gamma_j+i\delta_j)^2\right\}\cot\frac{\phi_{jr}+i\phi_{ji}}{2}, \tag{24}$$

where $j = 1, 2$. When expressing α_j, β_j, γ_j, and δ_j in terms of ϕ_j and θ_j as

$$\begin{cases} \alpha_j + i\beta_j = i\sqrt{2ru_0^2}\sin\frac{\phi_j}{2}\cos\theta_j, \\ \gamma_j + i\delta_j = i\sqrt{2ru_0^2}\sin\frac{\phi_j}{2}\sin\theta_j, \end{cases} \tag{25}$$

M_j and $L_j e^{i\varphi_j}$ are given by

$$M_j = \frac{\cosh\phi_{ji} + \cosh 2\theta_{ji}}{\cos\phi_{jr} + \cosh 2\theta_{ji}}, \tag{26}$$
$$L_1 e^{i\varphi_1} = \frac{\cos\frac{\phi_1-\phi_2}{2} - \cos(\theta_1-\theta_2)}{\cos\frac{\phi_1+\phi_2}{2} - \cos(\theta_1-\theta_2)}, \tag{27}$$
$$L_2 e^{i\varphi_2} = \frac{\cos\frac{\phi_1-\phi_2^*}{2} + \cos(\theta_1-\theta_2^*)}{\cos\frac{\phi_1+\phi_2^*}{2} + \cos(\theta_1-\theta_2^*)}, \tag{28}$$

where $\phi_j = \phi_{jr} + i\phi_{ji}$ and $\theta_j = \theta_{jr} + i\theta_{ji}$. Equations (27) and (28) are expressed as

$$L_1 e^{i\varphi_1} = \frac{\sin\frac{1}{2}\left(\frac{\phi_r^- + i\phi_i^-}{2} + \theta_r^- + i\theta_i^-\right)\cdot\sin\frac{1}{2}\left(\frac{\phi_r^- + i\phi_i^-}{2} - \theta_r^- - i\theta_i^-\right)}{\sin\frac{1}{2}\left(\frac{\phi_r^+ + i\phi_i^+}{2} + \theta_r^- + i\theta_i^-\right)\cdot\sin\frac{1}{2}\left(\frac{\phi_r^+ + i\phi_i^+}{2} - \theta_r^- - i\theta_i^-\right)}, \tag{29}$$
$$L_2 e^{i\varphi_2} = \frac{\cos\frac{1}{2}\left(\frac{\phi_r^- + i\phi_i^+}{2} + \theta_r^- + i\theta_i^+\right)\cdot\cos\frac{1}{2}\left(\frac{\phi_r^- + i\phi_i^+}{2} - \theta_r^- - i\theta_i^+\right)}{\cos\frac{1}{2}\left(\frac{\phi_r^+ + i\phi_i^-}{2} + \theta_r^- + i\theta_i^+\right)\cdot\cos\frac{1}{2}\left(\frac{\phi_r^+ + i\phi_i^-}{2} - \theta_r^- - i\theta_i^+\right)}, \tag{30}$$

with $\phi_r^{\pm} = \phi_{1r} \pm \phi_{2r}$, $\phi_i^{\pm} = \phi_{1i} \pm \phi_{2i}$, $\theta_r^{\pm} = \theta_{1r} \pm \theta_{2r}$, and $\theta_i^{\pm} = \theta_{1i} \pm \theta_{2i}$. Following the same way as the previous paper, it is seen that the interaction between two periodic soliton solitons yields a phase shift $\log|L_1 L_2|$ before and after the collision. The conditions of singular interactions are given from $|L_1 L_2 e^{i(\varphi_1+\varphi_2)}| \to \infty$ or $|L_1 L_2 e^{i(\varphi_1+\varphi_2)}| \to 0$ as

$$\begin{cases} \theta_{2i} = \pm\dfrac{\phi_{2i}}{2} + \left(\theta_{1i} \pm \dfrac{\phi_{1i}}{2}\right), & (31a) \\ \theta_{2r} = \pm\dfrac{\phi_{2r}}{2} + \left(\theta_{1r} \pm \dfrac{\phi_{1r}}{2}\right) + 2n_1\pi, & (31b) \end{cases}$$

$$\begin{cases} \theta_{2i} = \pm\dfrac{\phi_{2i}}{2} - \left(\theta_{1i} \pm \dfrac{\phi_{1i}}{2}\right), & (32a) \\ \theta_{2r} = \pm\dfrac{\phi_{2r}}{2} + \left(\theta_{1r} \pm \dfrac{\phi_{1r}}{2}\right) + (2n_2+1)\,\pi, & (32b) \end{cases}$$

or

$$\begin{cases} \theta_{2i} = \pm\dfrac{\phi_{2i}}{2} + \left(\theta_{1i} \mp \dfrac{\phi_{1i}}{2}\right), & (33a) \\ \theta_{2r} = \pm\dfrac{\phi_{2r}}{2} + \left(\theta_{1r} \mp \dfrac{\phi_{1r}}{2}\right) + 2n_3\pi, & (33b) \end{cases}$$

$$\begin{cases} \theta_{2i} = \pm\dfrac{\phi_{2i}}{2} - \left(\theta_{1i} \mp \dfrac{\phi_{1i}}{2}\right), & (34a) \\ \theta_{2r} = \pm\dfrac{\phi_{2r}}{2} + \left(\theta_{1r} \mp \dfrac{\phi_{1r}}{2}\right) + (2n_4+1)\,\pi, & (34b) \end{cases}$$

$$(n_1, n_2, n_3, n_4 = 0, \pm1, \pm2, \cdots)$$

respectively, where the conditions with suffixes (a) and (b) after the equation number are hereinafter called first condition and the second condition, respectively. Equations (31) and (32) are obtained by equating the denominators of $L_1e^{i\varphi_1}$ and $L_2e^{i\varphi_2}$ to zero, respectively. Equations (33) and (34) are obtained by equating the numerators of $L_1e^{i\varphi_1}$ and $L_2e^{i\varphi_2}$ to zero, respectively. The conditions shown by Equations (31) and (32) (obtained by equating the denominators of $L_1e^{i\varphi_1}$ and $L_2e^{i\varphi_2}$ to zero, respectively) are called resonant conditions and those shown by Equations (33) and (34) (obtained by equating the numerators of $L_1e^{i\varphi_1}$ and $L_2e^{i\varphi_2}$ to zero, respectively) are called conditions of the long-range interaction, temporarily. The lines of the first and second conditions are derived to the typical conditions of θ_{1r}, θ_{1i}, ϕ_{1r} and ϕ_{1i} in Figs. 1 and 2, respectively. The solid and broken line indicate resonant conditions and condition of the long-range interactions, respectively.

Taking the parameter in the vicinity of $\phi_{2r} = (2n+1)\pi$ and $\theta_{2i} = 0$, the soliton having the parameter of the suffix 2 becomes infinitesimal periodic soliton. It is understood from Figures 1 and 2 that in vicinity of $\phi_{2r} = (2n+1)\pi$ and $\theta_{2i} = 0$, the parameters that meet the resonant conditions or the condition of the long-range interactions do not exist except in the locations other than the points where the broken line and solid line cross each other (e.g. point S). Further, the following is obtained by substituting equation (14) into equations (29) and (30),

$$\begin{aligned} L_1L_2e^{i(\varphi_1+\varphi_2)} &= 1 - (\bar{\varepsilon}_1 + i\bar{\varepsilon}_2)\frac{\cos(A_2+B_1)}{\sin A_2 \cos B_1} \\ &\quad -(\bar{\varepsilon}_1 - i\bar{\varepsilon}_2)\frac{\cos(A_1+B_2)}{\sin A_1 \cos B_2} \\ &= 1 + O(\bar{\varepsilon}) \end{aligned}$$

resulting in $|L_1L_2| \to 1$ at the limit of $\bar{\varepsilon} \to 0$. From the above, it is understood that the phase shift that occurs in the case of collision of a periodic soliton and an infinitesimal periodic soliton turns out to be 0 and nothing occurs in the collision of a periodic soliton with the infinitesimal periodic soliton.

However, it is very interesting that there are intersections of a solid and a broken line in Figures 1 and 2. Moreover, the value of $L_1^2L_2^2$ at the point S becomes 0/0 and the interaction becomes sensitive to any small change in parameter in the vicinity of S. Now, we examine interactions between the periodic soliton and the infinitesimal periodic soliton with parameters of the point S in the vicinity. Let us calculate the values of parameters of the point S. When parameters of the first soliton are taken as $(\phi_{1r}, \phi_{1i}, \theta_{1r}, \theta_{1i}) = (\Phi, \Psi, \Theta, \Lambda)$, parameters of the point P are $(\phi_{2r}, \phi_{2i}, \theta_{2r}, \theta_{2i}) = (\pi, 2\Lambda - \Psi, \Theta - (\Phi + \pi)/2, 0)$. Now, we consider the following case,

$$\begin{cases} \phi_{2r} = \pi + 2\varepsilon_1, \\ \phi_{2i} = 2\Lambda - \Psi + 2\varepsilon_2, \\ \theta_{2r} = \Theta - \frac{\Phi+\pi}{2} + \varepsilon_3, \\ \theta_{2i} = \varepsilon_4, \end{cases} \tag{35}$$

where $O(\varepsilon_1) \sim O(\varepsilon_2) \sim O(\varepsilon_3) \sim O(\varepsilon_4) \sim O(\varepsilon)$ and $|\varepsilon| \ll 1$. Substituting equation (35) into equations (25), (26), (29), and (30), the followings are obtained

$$\alpha_2 = \sqrt{2ru_0^2}\left(\varepsilon_1 \sin\Theta^- \sinh\Lambda^- - \varepsilon_4 \cos\Theta^- \cosh\Lambda^-\right) + O(\varepsilon^2), \tag{36}$$

$$\gamma_2 = -\sqrt{2ru_0^2}\left(\varepsilon_1 \cos\Theta^- \sinh\Lambda^- + \varepsilon_4 \sin\Theta^- \cosh\Lambda^-\right) + O(\varepsilon^2), \tag{37}$$

$$\beta_2 = \sqrt{2ru_0^2}\left(\sin\Theta^- \cosh\Lambda^- + \varepsilon_2 \sin\Theta^- \sinh\Lambda^- + \varepsilon_3 \cos\Theta^- \cosh\Lambda^-\right) + O(\varepsilon^2), \tag{38}$$

$$\delta_2 = \sqrt{2ru_0^2}\left(-\cos\Theta^- \cosh\Lambda^- - \varepsilon_2 \cos\Theta^- \sinh\Lambda^- + \varepsilon_3 \sin\Theta^- \cosh\Lambda^-\right) + O(\varepsilon^2), \tag{39}$$

$$M_2 \simeq \frac{1 + \cosh(2\Lambda - \Psi)}{2(\varepsilon_1^2 + \varepsilon_4^2)}, \tag{40}$$

$$L_1 e^{i\varphi_1} \simeq -\frac{\sin\left(\frac{\Phi+i\Psi}{2}\right)\cosh\left(\Lambda - \frac{\Psi}{2}\right)}{\cos\left\{\frac{\Phi+2i\Lambda}{2}\right\}} \cdot \frac{1}{\sin\left\{\frac{(\varepsilon_1+\varepsilon_3)+i(\varepsilon_2+\varepsilon_4)}{2}\right\}}, \tag{41}$$

and

$$L_2 e^{i\varphi_2} \simeq -\frac{\cos\left\{\frac{\Phi+2i\Lambda}{2}\right\}}{\sin\left(\frac{\Phi+i\Psi}{2}\right)\cosh\left(\Lambda - \frac{\Psi}{2}\right)} \cdot \sin\left\{\frac{(\varepsilon_1 - \varepsilon_3) - i(\varepsilon_2 - \varepsilon_4)}{2}\right\}, \tag{42}$$

respectively. Thus, $L_1^2 L_2^2$ is given by

$$L_1^2 L_2^2 \simeq \frac{(\varepsilon_1 - \varepsilon_3)^2 + (\varepsilon_2 - \varepsilon_4)^2}{(\varepsilon_1 + \varepsilon_3)^2 + (\varepsilon_2 + \varepsilon_4)^2}. \tag{43}$$

Taking the parameters so that $(\varepsilon_1 - \varepsilon_3)^2 + (\varepsilon_2 - \varepsilon_4)^2 \approx (\varepsilon_1 + \varepsilon_3)^2 + (\varepsilon_2 + \varepsilon_4)^2$, we have $L_1^2 L_2^2 \approx 1$. Then, the interaction has no phase shift. However, in the case that $|\varepsilon_2 - \varepsilon_4|/|\varepsilon_2 + \varepsilon_4| \gg 1$ and $|\varepsilon_1 - \varepsilon_3|/|\varepsilon_1 + \varepsilon_3| \gg 1$, $L_1^2 L_2^2$ becomes large ($L_1^2 L_2^2 \gg 1$). For example, taking $\varepsilon_2 = -\varepsilon_4(1 + a\varepsilon_4)$ and $\varepsilon_3 = -\varepsilon_1(1 + b\varepsilon_1)$, where $a, b \sim O(1)$, the following is obtained

$$L_1^2 L_2^2 \sim O(\varepsilon^{-2}) \gg 1. \tag{44}$$

On the other hand, when we take the parameters so that $|\varepsilon_2 - \varepsilon_4|/|\varepsilon_2 + \varepsilon_4| \ll 1$ and $|\varepsilon_1 - \varepsilon_3|/|\varepsilon_1 + \varepsilon_3| \ll 1$, $L_1^2 L_2^2$ becomes infinitesimal. For example, taking $\varepsilon_2 = \varepsilon_4(1 + a'\varepsilon_4)$ and $\varepsilon_3 = \varepsilon_1(1 + b'\varepsilon_1)$, where $a', b' \sim O(1)$, the following is obtained

$$L_1^2 L_2^2 \sim O(\varepsilon^2) \ll 1. \tag{45}$$

Figures 3 and 4 show a schematic diagram of the world lines of the soliton hump in the x-y plane for the case $\alpha_1 > 0$, $\gamma_1 < 0$, $\alpha_2 > 0$, $\gamma_2 > 0$ and snapshot of the area bounded by the squares in the case of $L_1^2 L_2^2 \gg 1$ and $L_1^2 L_2^2 \ll 1$, respectively. The difference in parameters in the two figures is small. However, the phenomenon is different in a dramatic form. It is understood that the interaction with parameters in the region in the vicinity of the point S is parameter sensitive.

4 Conclusion

It has been shown that there exists a very small amplitude periodic soliton which interacts resonantly with finite amplitude periodic soliton under certain conditions. Although the existence of resonance phenomena is interesting, it is more interesting that these interactions always become parameter-sensitive. It is thought that these originate in a mathematical principle feature of a dispersion relation of DS II equation.

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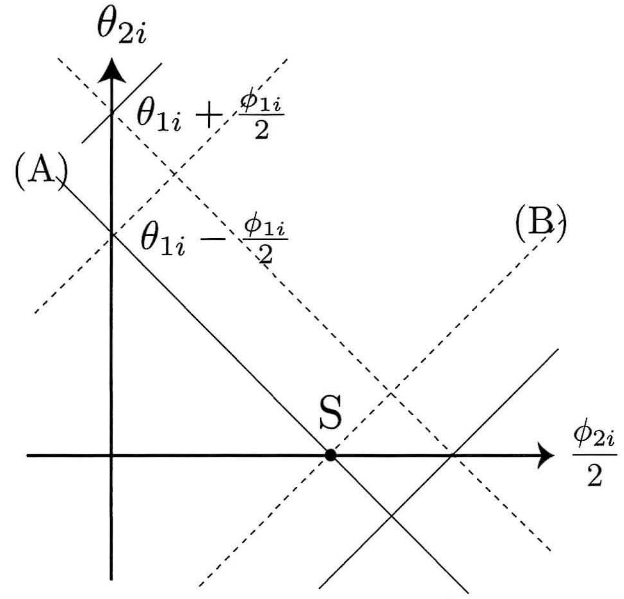

Figure 1: Cross section of the parameter space $(\phi_{1i}, \phi_{2i}, \theta_{1i}, \theta_{2i})$ related to the first conditions for the singular interaction between two periodic solitons. The figure is drawn for fixed θ_{1i} and ϕ_{1i}. The first conditions for resonance and the long-range interaction are satisfied on the solid lines and the broken lines, respectively. Line (A) is $\theta_{2i} = -\phi_{2i}/2 + (\theta_{1i} - \phi_{1i}/2)$ (It is given from equation (31a)). Line (B) is $\theta_{2i} = \phi_{2i}/2 - (\theta_{1i} - \phi_{1i}/2)$ (It is given from equaiton (34a)).

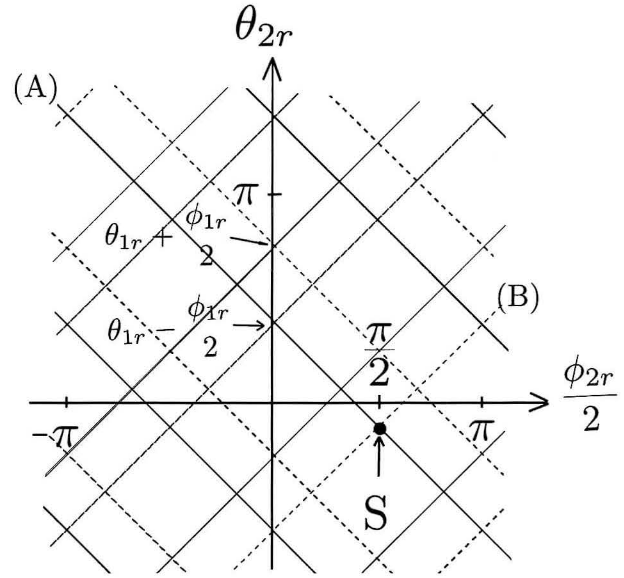

Figure 2: Cross section of the parameter space $(\phi_{1r}, \phi_{2r}, \theta_{1r}, \theta_{2r})$ related to the second conditions for the singular interaction between two periodic solitons. The figure is drawn for fixed θ_{1r} and ϕ_{1r}. The first conditions for resonance and the long-range interaction are satisfied on the solid lines and the broken lines, respectively. Line (A) is $\theta_{2r} = -\phi_{2r}/2 + (\theta_{1r} - \phi_{1r}/2)$ (It is given from equation (31b) with $n_1 = 0$). Line (B) is $\theta_{2r} = \phi_{2r}/2 + (\theta_{1r} - \phi_{1r}/2) - \pi$ (It is given from equaiton (34b) with $n_4 = -1$).

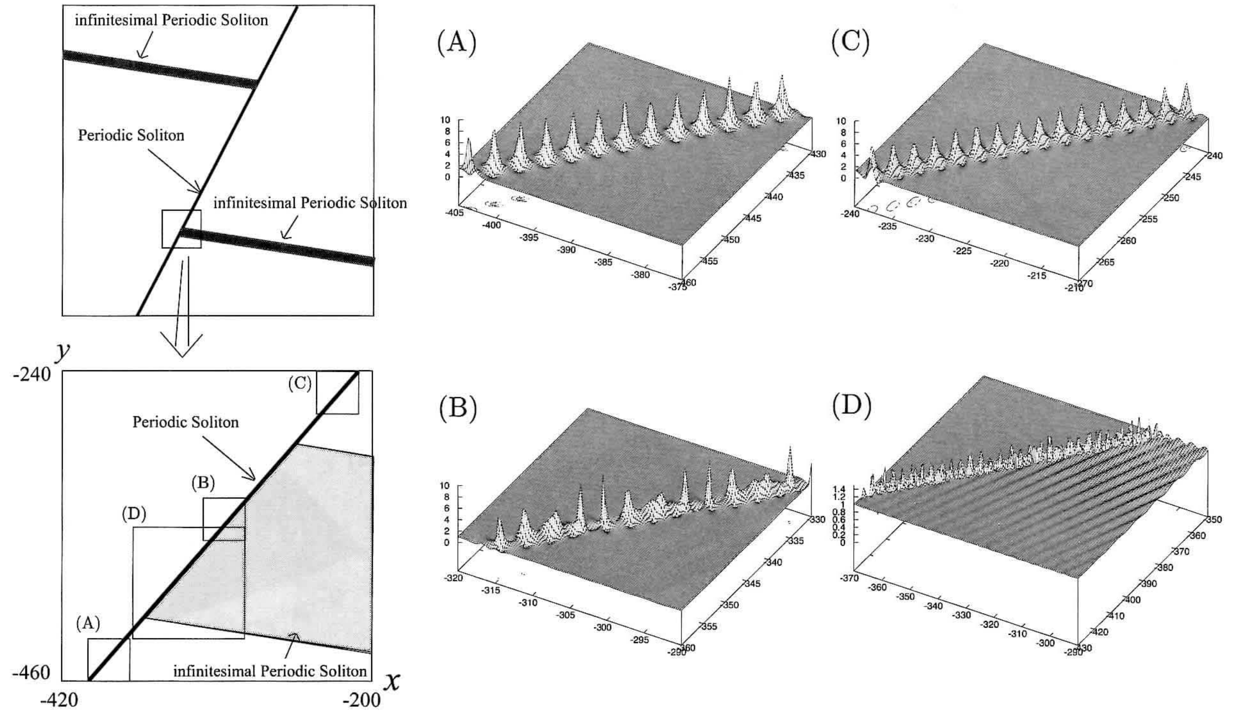

Figure 3: A schematic diagram of the world lines of the soliton hump in the x-y plane for the case $\alpha_1 > 0$, $\gamma_1 < 0$, $\alpha_2 > 0$, $\gamma_2 > 0$ and a snapshot of the area bounded by the squares in the case of $L_1^2 L_2^2 \gg 1$. The parameters are as follows: $\Phi = (3/8)\pi$, $\Psi = 1.6$, $\Theta = (9/16)\pi$, $\Lambda 1.0$, $\varepsilon_1 = 0.02$, $\varepsilon_2 = -0.02001$, $\varepsilon_3 = -0.02001$, and $\varepsilon_4 = 0.02$. The values of $L_1 L_2$ is 4.05×10^3.

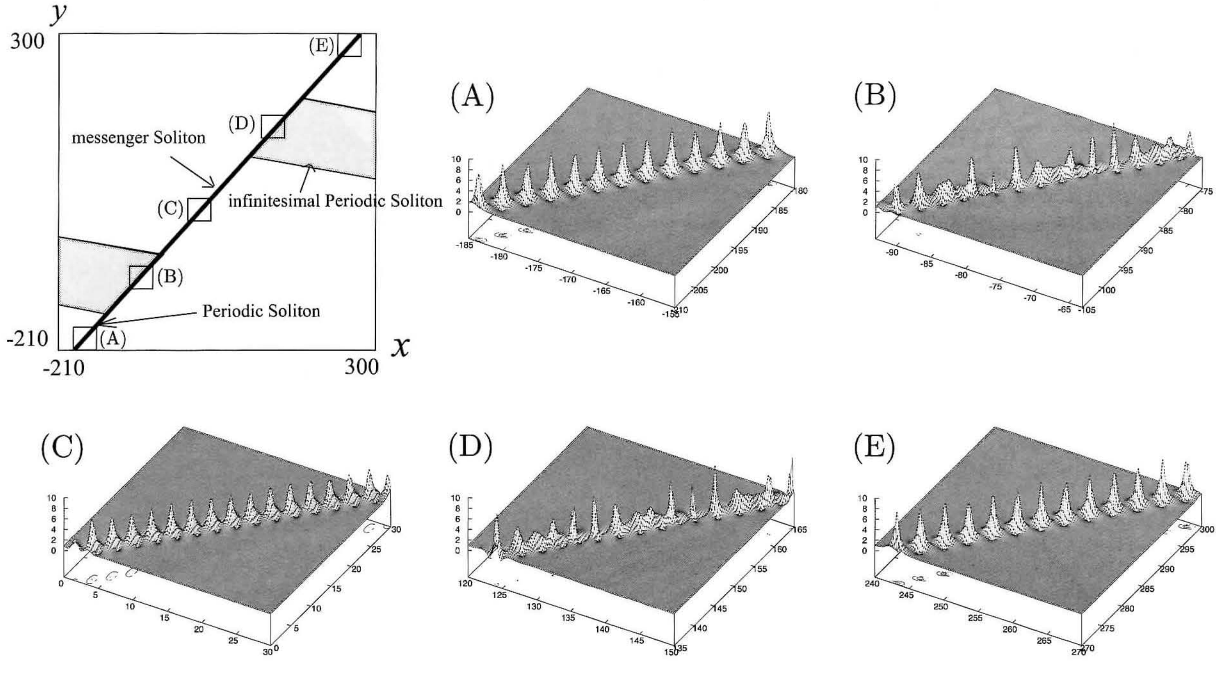

Figure 4: A schematic diagram of the world lines of the soliton hump in the x-y plane for the case $\alpha_1 > 0$, $\gamma_1 < 0$, $\alpha_2 > 0$, $\gamma_2 > 0$ and a snapshot of the area bounded by the squares in the case of $L_1^2 L_2^2 \gg 1$. The parameters are as follows: $\Phi = (3/8)\pi$, $\Psi = 1.6$, $\Theta = (9/16)\pi$, $\Lambda 1.0$, $\varepsilon_1 = -0.02$, $\varepsilon_2 = -0.02001$, $\varepsilon_3 = -0.02001$, and $\varepsilon_4 = -0.02$. The values of $L_1 L_2$ is 2.53×10^{-4}.