Resonant interaction between two y-periodic solitons: Solutions to the Davey-Stewartson I equation

Takahito Arai
Research Institute for Science and Technology,
Kinki University, Higashi-Osaka 577-8502, JAPAN

(Received 11 January, 2011)

Abstract

The exact solutions to the Davey-Stewartson I equation are analyzed to study the interactions between two y-periodic solitons. There are two types of singular interactions; one is the resonant interaction that generates one periodic soliton after collision and the other is the long-range interaction where two solitons interchange each other infinitely apart.

Key words: Davey-Stewarton equation, periodic soliton, soliton resonance

1 Introduction

The Davey-Stewartson I (DS I) equation may be written as [1]

$$\begin{cases} iu_t + u_{xx} + u_{yy} + r|u|^2u - 2uv = 0, \\ v_{xx} - v_{yy} - r(|u|^2)_{xx} = 0, \end{cases} \tag{1}$$

where r is constant. It is well known that this equation is the two-dimensional generalization of the nonlinear Schrödinger equation. The solutions to the DS I equation has been studied previously in various aspect. The N-soliton solution was obtained by Anker and Freeman in the inverse scattering method [2] and by Satsuma and Ablowitz in Hirota method [3]. The interaction between two line solitons has been studied by Anker and Freeman, and they found the existence of soliton resonance [4]. It is known that the DS I equation has not only line soliton solutions which have essentially one-dimensional structures but also various soliton solutions which have the structures peculiar to high-dimensionality. Over the past many years, several studies have been made on interactions between various kinds of nonlinear waves of the DS I equation [5, 6, 7, 8, 9, 10]. We found the existence of the resonant interactions between two periodic solitons and between the periodic soliton and other types of solitons, which are essentially different from the resonant interaction between two line solitons. We expect that it is very important to study the interactions of nonlinear waves in discussing the dynamics of the unstable wave field. In fact, Tajiri *et al* and Pelinovsky have pointed out that the existence of the periodic soliton resonance may be related to the instability of solitons [7, 11, 12, 13]. The purposes of this paper are (i) to review the y-periodic soliton solution and investigate the interaction between two y-periodic solitons and (ii) to show that there are two types of singular interactions.

2 The interactions between two y- periodic solitons

2.1 y-periodic soliton solution

The one y-periodic soliton solution of the DS I equation is given by

$$u = \frac{g}{f}, \quad v = -2(\log f)_{xx}, \tag{2}$$

with

$$f(\xi', \eta) = 1 + e^{\xi'} \cos \eta + \frac{M}{4} e^{2\xi'} \tag{3}$$

$$g(\xi', \eta, \phi) = u_0 e^{i\zeta} f(\xi' + i\phi, \eta) \tag{4}$$

where

$$\begin{aligned}
&\zeta = kx + ly - \omega t + \zeta^0, \\
&\xi' = \alpha x - \Omega t + \xi^0, \\
&\eta = \delta y - \gamma t + \eta^0, \\
&\sin^2 \frac{\phi}{2} = \frac{\alpha^2 + \delta^2}{2ru_0^2},
\end{aligned} \tag{5}$$

$$\begin{aligned}
\Omega + i\gamma &= 2k\alpha + 2il\delta \\
&\quad - (\alpha^2 - \delta^2) \cot \frac{\phi}{2},
\end{aligned} \tag{6}$$

$$M = \frac{-4\delta^2}{4\alpha^2 - 2ru_0^2 \sin^2 \phi}. \tag{7}$$

Substituting equations (3) and (4) into equation (2), we see that u and v are expressed as

$$u = u_0 e^{i\zeta} \frac{\cosh(\xi + i\phi) + \dfrac{1}{\sqrt{M}} \cos \eta}{\cosh \xi + \dfrac{1}{\sqrt{M}} \cos \eta}, \tag{8}$$

$$v = -2\alpha^2 \frac{1 + \dfrac{1}{\sqrt{M}} \cosh \xi}{\left\{ \cosh \xi + \dfrac{1}{\sqrt{M} \cos \eta} \right\}^2}, \tag{9}$$

where $\xi = \xi' + \log(\sqrt{M}/2)$. From (8) and (9), we see that the existence condition for nonsingular solution is given by $M > 1$. If we express the wave numbers α and δ in terms of ϕ and θ as

$$\alpha = \sqrt{2ru_0^2} \sin \frac{\phi}{2} \cos \theta, \tag{10}$$

$$\delta = \sqrt{2ru_0^2} \sin \frac{\phi}{2} \sin \theta, \tag{11}$$

then equation (7) is rewritten as

$$M = \frac{1 - \cos 2\theta}{\cos \phi - \cos 2\theta}. \tag{12}$$

From equation (12), the existence condition for the non-singular solution (8) and (9) is give by

$$\cos \phi > \cos 2\theta. \tag{13}$$

2.2 Interactions between two y-periodic solitons

Now, we consider the phase shift of two y-periodic solitons due to the interaction. The solution describing the interaction between two y-periodic solitons is given by equation (2) with

$$\begin{aligned}
f(\xi_1, \xi_2, \eta_1, \eta_2) = 1 &+ \frac{M_1}{4} e^{2\xi_1} + \frac{M_2}{4} e^{2\xi_2} + \frac{M_1 M_2 N_1^2 N_2^2}{16} e^{2(\xi_1 + \xi_2)} \\
&+ e^{\xi_1} \left(1 + \frac{M_2 N_1 N_2}{4} e^{2\xi_2} \right) \cos \eta_1 + e^{\xi_2} \left(1 + \frac{M_1 N_1 N_2}{4} e^{2\xi_1} \right) \cos \eta_2 \\
&+ \frac{1}{2} e^{\xi_1 + \xi_2} \left\{ N_1 \cos(\eta_1 + \eta_2) + N_2 \cos(\eta_1 - \eta_2) \right\},
\end{aligned} \tag{14}$$

$$g(\xi_1,\xi_2,\eta_1,\eta_2,\phi_1,\phi_2) = u_0 e^{i\zeta} f(\xi_1+i\phi_1,\xi_2+i\phi_2,\eta_1,\eta_2), \tag{15}$$

where

$$\xi_j = \alpha_j x - \Omega_j t + \xi_j^0, \quad \eta_j = \delta_j y - \gamma_j t + \eta_j^0,$$

$$\sin^2\frac{\phi_j}{2} = \frac{\alpha_j^2+\delta_j^2}{2ru_0^2}, \tag{16}$$

$$\Omega_j + i\gamma_j = 2k\alpha_j + 2il\delta_j - \left(\alpha_j^2 - \delta_j^2\right)\cot\frac{\phi_j}{2}, \tag{17}$$

$$M = \frac{-4\delta^2}{4\alpha^2 - 2ru_0^2\sin^2\phi}. \tag{18}$$

$$N_1 = -\frac{(\alpha_1-\alpha_2)^2+(\delta_1-\delta_2)^2-2ru_0^2\sin^2\dfrac{\phi_1-\phi_2}{2}}{(\alpha_1+\alpha_2)^2+(\delta_1+\delta_2)^2-2ru_0^2\sin^2\dfrac{\phi_1+\phi_2}{2}}, \tag{19}$$

$$N_2 = -\frac{(\alpha_1-\alpha_2)^2+(\delta_1+\delta_2)^2-2ru_0^2\sin^2\dfrac{\phi_1-\phi_2}{2}}{(\alpha_1+\alpha_2)^2+(\delta_1-\delta_2)^2-2ru_0^2\sin^2\dfrac{\phi_1+\phi_2}{2}}. \quad (j=1,2) \tag{20}$$

If we express α_j and δ_j in terms of ϕ_j and θ_j as

$$\alpha_j = \sqrt{2ru_0^2}\sin\frac{\phi_j}{2}\cos\theta_j, \tag{21}$$

$$\delta_j = \sqrt{2ru_0^2}\sin\frac{\phi_j}{2}\sin\theta_j, \tag{22}$$

M_j, N_1 and N_2 are rewritten as

$$M_j = \frac{1-\cos 2\theta_j}{\cos\phi_j - \cos 2\theta_j}, \tag{23}$$

$$N_1 = \frac{\cos\dfrac{\phi_1-\phi_2}{2} - \cos(\theta_1-\theta_2)}{\cos\dfrac{\phi_1+\phi_2}{2} - \cos(\theta_1-\theta_2)}, \tag{24}$$

$$N_2 = \frac{\cos\dfrac{\phi_1-\phi_2}{2} - \cos(\theta_1+\theta_2)}{\cos\dfrac{\phi_1+\phi_2}{2} - \cos(\theta_1+\theta_2)}, \tag{25}$$

respectively. The existence conditions for the non-singular solution are given by

$$\cos\phi_j > \cos 2\theta_j, \tag{26}$$

which is obtained by $M_j > 1$.

We consider the phase shift after the collision by using equation (14). Now, we assume that $\alpha_1 > 0$, $\alpha_2 > 0$ and $\Omega_1/\alpha_1 > \Omega_2/\alpha_2$. From equation (14), the two separated solitons long before and after the interaction are given by

$$f_1 = 1 + e^{\xi_1}\cos\eta_1 + \frac{M_1}{4}e^{2\xi_1},$$

$$f_2 = \frac{M_1}{4}e^{2\xi_1}\left(1 + N_1N_2e^{\xi_2}\cos\eta_2 + \frac{M_2N_1^2N_2^2}{4}e^{2\xi_2}\right),$$

and

$$f_1 = \frac{M_2}{4}e^{2\xi_2}\left(1 + N_1N_2e^{\xi_1}\cos\eta_1 + \frac{M_1N_1^2N_2^2}{4}e^{2\xi_1}\right),$$

$$f_2 = 1 + e^{\xi_2}\cos\eta_2 + \frac{M_2}{4}e^{2\xi_2},$$

respectively, where subscript 1 and 2 of f denote the each y-periodic soliton. Taking into account that u and v are unchanged even if f and g multiplied by $\exp(ax+b)$ with a and b independent of x, we find that the interaction is the form

$$\begin{aligned}&\left[f_1(\xi_1,\eta_1), g_1(\xi_1,\eta_1,\phi_1)\right], \\ &\qquad \left[f_2(\xi_2+\Delta,\eta_2), g_2(\xi_2+\Delta,\eta_2,\phi_2)\right] \\ \to\ &\left[f_1(\xi_1+\Delta,\eta_1), g_1(\xi_1+\Delta,\eta_1,\phi_1)\right], \\ &\qquad \left[f_2(\xi_2,\eta_2), g_2(\xi_2,\eta_2,\phi_2)\right],\end{aligned} \tag{27}$$

where $\Delta = \log |N_1 N_2|$. It is revealed that the interaction between two y-periodic solitons yields the after interaction solitons with a phase shift Δ. It is noted that the interaction is attractive or repulsive in the x-direction in the case $\Delta > 0$ ($|N_1 N_2| > 1$) or $\Delta < 0$ ($|N_1 N_2| < 1$), respectively.

3 Periodic soliton resonances

In this section, we consider the two singular interactions, one is the resonant interaction, the other is the long-range interaction. And we show the sequence of the snapshots of each singular interactions between two y-periodic solitons.

3.1 The resonant interaction

At first, we consider the condition $|N_1 N_2| \to \infty$. The phase shift Δ in the propagation direction becomes ∞. In the case $\alpha_1 \alpha_2 > 0$, this means that the two periodic solitons collide, propagate together and the period of the intermediate state persist infinity. This is thought as a resonant interaction between two y-periodic solitons. This conditions are obtained by setting the denominator of N_1 or N_2 to 0;

$$\cos \frac{\phi_1 + \phi_2}{2} = \cos(\theta_1 - \theta_2), \qquad (28)$$

or

$$\cos \frac{\phi_1 + \phi_2}{2} = \cos(\theta_1 + \theta_2). \qquad (29)$$

The sequence of snapshots of figure 1 shows the resonant interaction between two y-periodic solitons with parameters near the condition (28). Initially, two y-periodic solitons separated well enough to look like two independent solitons (figure 1(a)). When they collide each other, the interaction yields intermediate y-periodic soliton (figure 1(d)) and this quasi-resonant state persists over comparatively long period. It is noted that these resonant conditions correspond to the dispersion relations of the resonant periodic soliton.

3.2 The long-range interaction

Next, we consider the condition $|N_1 N_2| \to 0$. Two y-periodic solitons can interact infinitely apart each other, because the phase shift $\Delta \to -\infty$. This is thought as the long-range interaction between two y-periodic solitons. This conditions are obtained by setting the numerator of N_1 or N_2 to 0;

$$\cos \frac{\phi_1 - \phi_2}{2} = \cos(\theta_1 - \theta_2), \qquad (30)$$

or

$$\cos \frac{\phi_1 - \phi_2}{2} = \cos(\theta_1 + \theta_2). \qquad (31)$$

Graphical representations of interactions between two y-periodic solitons having parameter near the condition (30) are shown in figure 2. The solitons in figure 2(a) are the two y-periodic solitons before interaction. When the y-periodic soliton 1 (the left y-periodic soliton) approaches to the y-periodic soliton 2 (the right y-periodic soliton), the y-periodic soliton 1 receives a small transverse disturbance of the same wave number δ_2. The disturbance grows as two solitons approach. The y-periodic soliton 1 emits the y-periodic soliton forward (which is called a messenger soliton), and then changes into the periodic soliton 2 (figure 2(b) and 2(c)). When the messenger soliton collides with the y-periodic soliton 2, they interact resonantly to yield the new periodic soliton which structure is the same as the y-periodic soliton 1 before emitting the messenger soliton. It is noted that the messenger soliton and the y-periodic soliton 2 satisfy the condition of a resonant interaction.

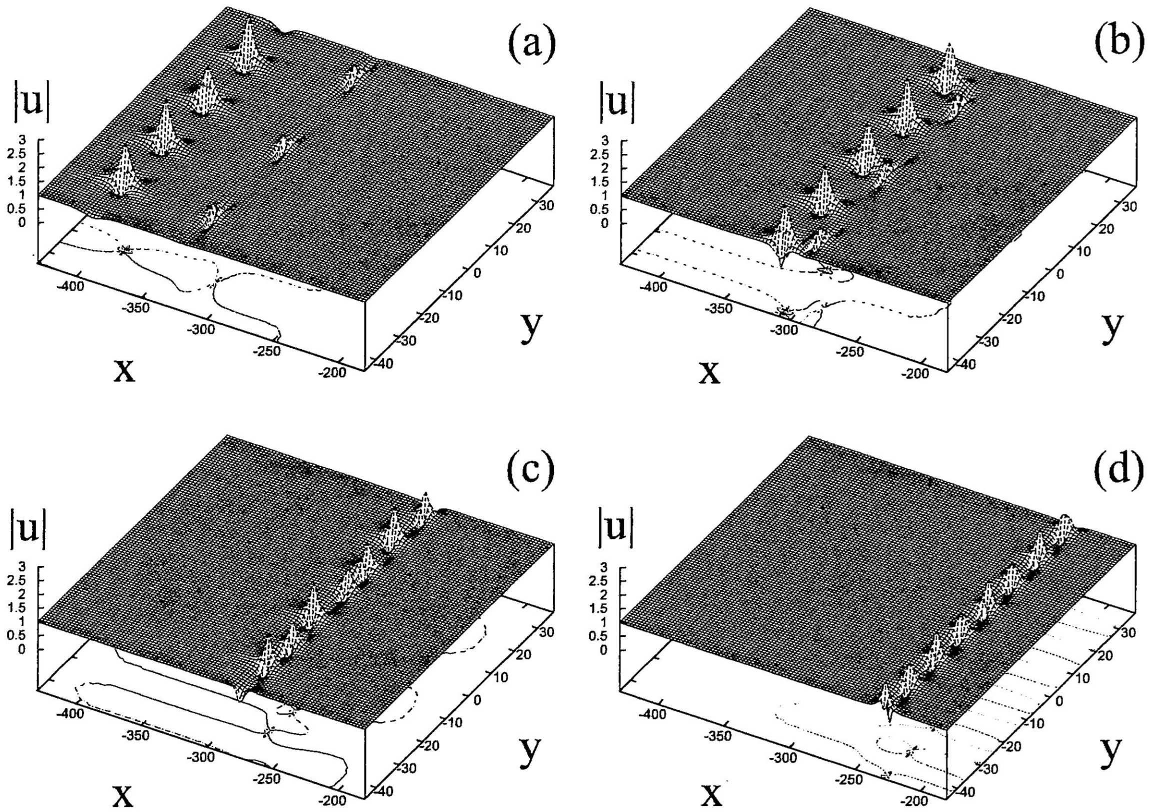

Figure 1: The sequence of snapshots of the resonant interaction between two y-periodic solitons. The parameters are $\phi_1 = \pi/6$, $\theta_1 = 5\pi/12$ and $\phi_2 = \pi/8$, $\theta_1 = 13\pi/48$. $N_1N_2 = 3.2 \times 10^{14}$. The times for the snapshots are (a) $t = -60$, (b) $t = -48$, (c) $t = -42$ and (d) $t = -30$. In this figure, x, y, and $|u|$ are all dimensionless.

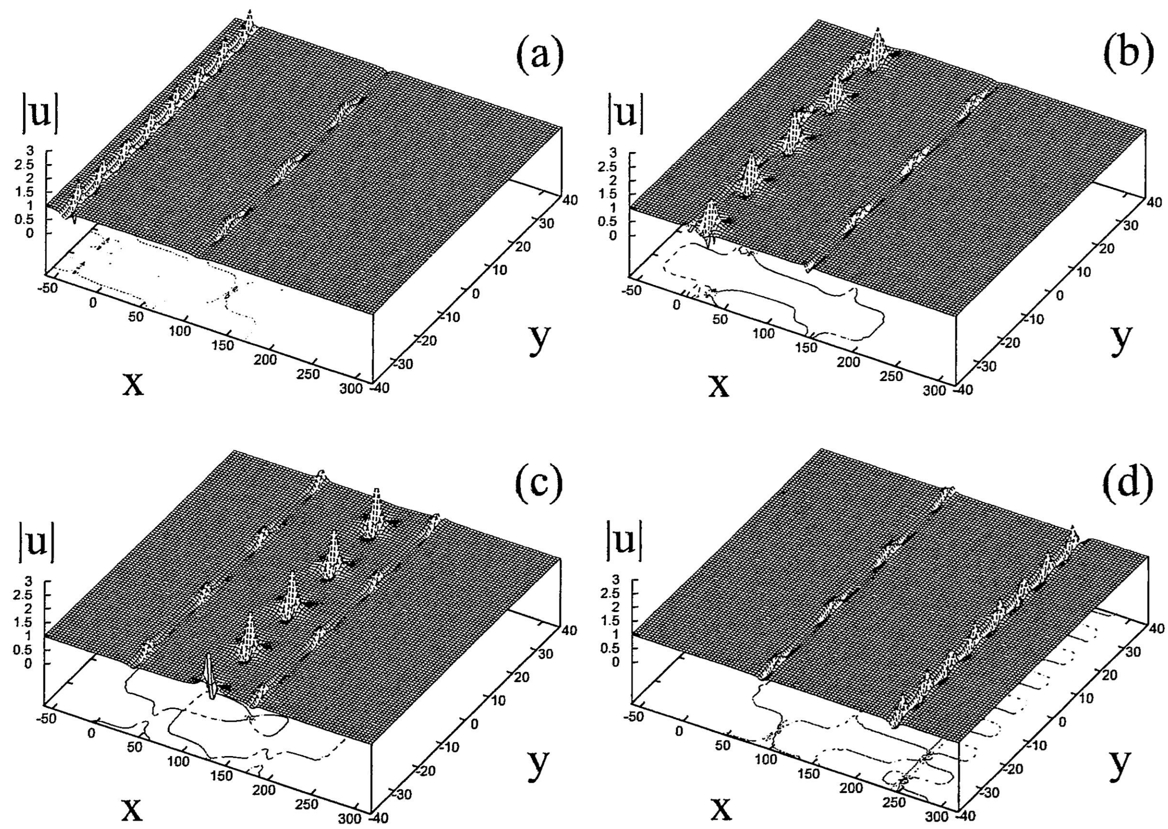

Figure 2: The sequence of snapshots of the long-range interaction between two y-periodic solitons. The parameters are $\phi_1 = \pi/3$, $\theta_1 = \pi/3$ and $\phi_2 = \pi/6$, $\theta_2 = \pi/4$. $N_1 N_2 = 3.0 \times 10^{-16}$. The times for the snapshots are (a) $t = -12$, (b) $t = 4$, (c) $t = 20$ and (d) $t = 44$. In this figure, x, y, and $|u|$ are all dimensionless.

4 Conclusion

We have investigated the interaction between two y-periodic solitons and shown that there are two types of singular interactions. One is the resonant interaction where two y-periodic soliton interact so as to make a new y-periodic soliton, the other is the long-range interaction where two y-periodic soliton interact each other infinitely apart through the periodic soliton (which is called a messenger soliton). If α and δ are taken as complex number, we can examine the interaction between two inclined periodic solitons. Recently, it has been demonstrated the existence of the long-range interaction between two periodic solitons through the growing-and decaying mode [10].

Finally, it is pointed out that the periodic soliton resonance is not peculiar to the DS I equation. The existences of periodic soliton resonance are shown to the Kadomtsev-Petviashvili equation with positive dispersion and DS II equation.

References

[1] A. Davey and K. Stewartson (1974) Proc. R. Soc. London A **338**, 101.

[2] D. Anker and N. C. Freeman (1978) Proc. R. Soc. London A **360**, 529.

[3] J. Satsuma and M. J. Ablowitz (1979) J. Math. Phys. **20**, 1496.

[4] D. Anker and N. C. Freeman (1978) J. Fluid Mech. **87**, 17.

[5] Y. Watanabe and M. Tajiri (1998) J. Phys. Soc. Jpn. **67**, 705.

[6] M. Tajiri, T. Arai and Y. Watanabe (1998) J. Phys. Soc. Jpn. **67**, 4051.

[7] D. E. Pelinovsky (1994) Phys. Lett. A **196**, 181.

[8] M. Tajiri, K. Takeuchi and T. Arai (2001) Phys. Rev. E **64**, 056622.

[9] M. Tajiri, H. Miura and T. Arai (2002) Phys. Rev. E **66**, 067601.

[10] T. Arai and M. Tajiri (2010) J. Phys. Soc. Jpn. **79**, 045002.

[11] M. Tajiri and Y. Murakami (1992) Kokyuroku, *Mathematical Aspect on Nonlinear Waves in Fluids*, Res. Inst. Math. Sci. Kyoto Univ. **782**, 206 [in Japanese].

[12] M. Tajiri, Y. Murakami and K. Kimura (1994) Bull. Univ. Osaka Pref. Ser. A **43**, 55.

[13] M. Tajiri, K. Takeuchi and T. Arai (2001) J. Phys. Soc. Jpn. **70**, 1505.