Energy Variability and Chaos in Ueda Oscillator

Bharti[1], L. M. SAHA[2] *and* Manabu YUASA[3]
[1]*Shyam Lal College(Evening), University of Delhi, Delhi 110032, INDIA*
[2]*Mathematical Sciences Foundation, N-91, Greater Kailash I, New Delhi-110048, INDIA*
[3]*Research Institute for Science and Technology, Kinki University, Higashi-Osaka 577-8502, JAPAN*

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Abstract

Use of Melnikov method enables us to prove the existence of transverse homoclinic points and homoclinic bifurcations occurring in a number of dynamical systems. Energy plays a crucial role in the description of the evolutionary behaviour of nonlinear dynamical systems. If the energy behaves like a variable; it leads us to an interesting way to understand results obtained in investigation of various systems. Melnikov function, that measures the distance between the stable and unstable manifolds of the saddle equilibrium of the Poincaré map of sections near the separatrix, has been associated with the energy variable of the Hamiltonian. The system used here is the Ueda oscillator, [3, 5, 12, 13], which displays very interesting results during evolution shown through these works. We have again investigated the Ueda oscillator and studied its chaotic and transient chaotic evolutions taking into account the concept of energy variability. With the adjustment of certain parameter, we have observed, the chaotic, transient chaotic and regular behaviour through phase plots and Poincaré surface of sections. Numerical results are obtained to support the analytical calculations which are discussed through various graphics.

Key words: Energy variability, Chaotic dynamical system, Ueda oscillator

1 Introduction

During the past few decades, chaotic behaviour has been observed in a wide range of dynamical systems governed by simple deterministic equations [6, 10]. With the extensive studies of typical nonlinear systems, it has been revealed that the chaotic motion emerging in various nonlinear systems can be suppressed or synchronized. In the chaotic regime two nearby trajectories diverge exponentially until they become completely uncorrelated and future prediction becomes inaccurate. There are some recent attempts where chaos has been used profitably by synchronizing chaotic orbits [11]. But at most of the places, chaos is an undesirable phenomenon which leads to unpredictable and violent vibrations. Practically, one would like to control the system dynamics with minimum efforts so that whenever chaos seems to be harmful, it can be changed to a desired periodic or fixed point attractor [2]. The concept of energy variability in nonlinear dynamical systems has been introduced by Ali [4] and he has applied this concept to DVP oscillator. Later it has been used for BVP oscillator and double well oscillator [1, 7, 8]. For a long time, we have been taking the nonlinear dynamical systems with constant energy. But, it seems, in studying the real phenomenon; the assumption of energy like a variable may provide significant explanation of the system behaviour. Thus, the exhibition of irregular and chaotic motions in a number of nonlinear systems strongly linked to the energy variability of the systems. Therefore, for the enhancement of investigation of the nonlinear dynamical systems, one may proceed with variable energy. This may provide interesting results.

Objective of this paper is to investigate the chaotic and regular behaviour in Ueda oscillator with variable energy. Also, our aim is to find the emergence of transient chaos in this oscillator as well as the duration of transient chaos. The analytical approach has been supported with numerical results displayed through graphics like phase plots and Poincaré surface section. Energy variable for a perturbed Hamiltonian system be defined in association with the Melnikov function which may decide in more interesting way the regular and chaotic behaviour. We define the energy variable for a perturbed Hamiltonian system as;

$$\frac{dx}{dt} = F(x) + \epsilon f(x,t), \tag{1}$$

where$x = [x_1, x_2]^T \in \mathcal{R}^2$, F: $\mathcal{R}^2 \longrightarrow \mathcal{R}^2$ and $F \equiv [\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x}]^T$ for some function $H(x)$. The perturbed term, (the second term) on the right hand side is small, i.e., $0 < \epsilon << 1$, and periodic with respect to the time variable, i.e. $f(x, t+\tau) = f(x,t)$ for any $x \in \mathcal{R}^2$ and $t \geqq 0$. When $\epsilon = 0$, we obtain the original unperturbed Hamiltonian system which is assumed to have a hyperbolic saddle equilibrium at the origin $(0, 0)$, and to possess a first integral, namely

$$H(x) = C. \tag{2}$$

The curves given by above equation specify all possible trajectories $\gamma(C) = \{x \in \mathcal{R}^2; H(x) = C\}$ of system

$$\frac{dx}{dt} = F(x) \tag{3}$$

in the phase-plane for different values of the energy constant C relative to different initial conditions. One and only one trajectory (the separatrix) among those can pass through the origin as (2). We may choose the Hamiltonian H so that the separatrix is given by the equation $H(x) = 0$. This separatrix divides the phase plane of system (3) into two main regions, namely $\gamma_+ = \{\gamma(C); C > 0\}$ and $\gamma_- = \{\gamma(C); C < 0\}$ [1, 4].

Melnikov function provide the measure of the distance between the stable and unstable manifolds of the saddle equilibrium of the Poincaré map of sections near the separatrix. A first order approximation of ϵ, of the Melnikov function can be given by [9],

$$M(t_0) = \int_{-\infty}^{+\infty} F(q_0(t-t_0)) \wedge f(q_0(t-t_0), t)dt, \tag{4}$$

where $q_0(t - t_0)$ is the separatrix of the Hamiltonian system (3). A simple zero of this function guarantee the existence of homoclinic points and hence, the transversal intersection of the stable and unstable manifolds. Smale-Birkhoff theorem asserts that when such transversal intersection occurs then some iterate of the Poincaré map has an invariant hyperbolic set.

As the system (1) is in perturbed form, it would be justified if we think of equation (2) as an integral of system (1), but with an exception that the energy level C for this system becomes a variable throughout the course of motion (i.e. $C \equiv C(x,t)$). Hence, a point in the phase plane of system (1) will be considered as moving from one energy level of system (3) to another level. Thus, we can predict the process of gaining or loosing energy, in the course of time, to be very slow (small) and proportional to the perturbation term in system (1).

Accordingly, considering equation (2) to be an integral of system (1) and by differentiating (2) with respect to time and substituting from (1), we obtain

$$\frac{dC}{dt}(x,t) = \epsilon\{F(x) \wedge f(x,t)\}. \tag{5}$$

This leads us to obtain the final form of the Melnikov function,

$$M(t_0) = \int_{-\infty}^{+\infty} \frac{dC}{dt}(q_0(t - t_0), t). \quad (6)$$

Results obtained by the application of Melnikov function enable us to decide the integrability and non-integrability of the equation of motion (1.1).

2 Ueda oscillator

2.1 Origin:

The origin of the Ueda oscillator can be explained in the following way:
Driven pendulums display some of the most significant examples of chaos and regularity. A pendulum can be mounted on a cart that oscillates periodically back and forth, driven by a variable-speed motor. These examples can be understood by assuming the motion results from a restoring force F proportional to $-\sin x$, a friction term proportional to $-\frac{dx}{dt}$, and a periodic driven force $A \sin \omega t$, which when substituted into Newton's second law $\left(F = m\frac{d^2x}{dt^2}\right)$ leads to the equation:

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \sin x = A \sin \omega t. \quad (7)$$

Ueda[13] studied a variant of this system with the $\sin x$ restoring force replaced by x^3 . The modified equation of oscillator, called Ueda oscillator, which is named after Y. Ueda, with the periodic current $c \sin \omega t$ along with a bias c be written as:

$$\frac{dx}{dt} = y \quad (8)$$

$$\frac{dy}{dt} = -ax^3 - by + c \sin \omega t \quad (9)$$

The Ueda system can be considered as a special case of Duffing ' s oscillator[14] that has both a linear and cubic restoring force, usually of opposite signs. Ueda oscillator can be assumed as a biologically and physically important dynamical model exhibiting chaotic motion. It can be used to explore much physical behaviour in biological systems.

2.2 Energy Variability Approach:

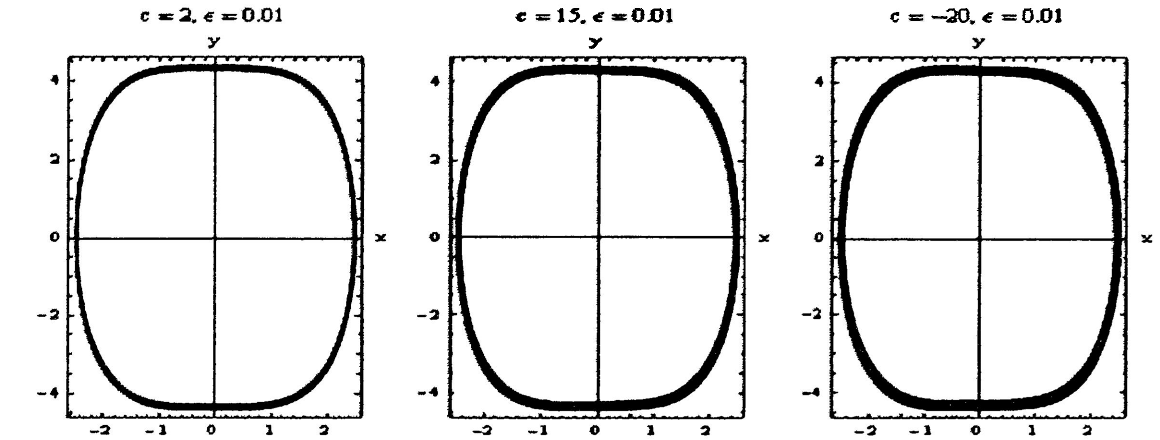
The Ueda oscillator defined in equations (8)and(9) is an oscillator with constant energy. With the use of certain results of energy variability, this oscillator can be transformed to a new form, where energy behaves as a variable and can be written as:

$$\frac{dx}{dt} = y \quad (10)$$

$$\frac{dy}{dt} = -ax^3 + \epsilon(-by + c \sin \omega t), \quad (11)$$

where b is the friction coefficient, and c is the strength of the driving force which oscillates at a frequency ω. The presence of ϵ indicates that we are dealing with variable energy. We will see that a transition to chaos now occurs as the strength of the driving force and variable energy.
We proceed now to investigate Ueda oscillator with variable energy as given in equations (10) and (11). For this we follow the following steps:

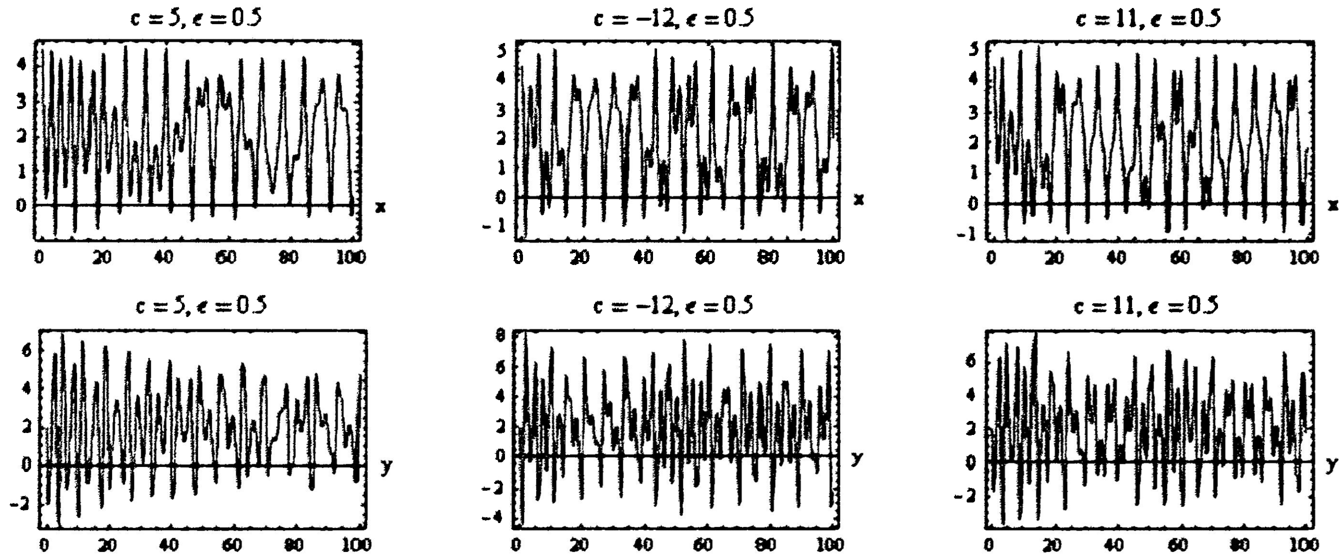
To perform numerical studies, we fix the parameter values as $a = 1; b = 0.06; \omega = 1$; and the initial values of x and y as $x_0 = 2.5$ and $y_0 = 0$. Then, we perform the numerical simulation to obtain the chaotic, regular and transient chaotic behaviour of the oscillator by varying the parameters c, which represent the strength of the driven force, and ϵ, which is due to the variation of energy. First we take the value of $\epsilon = 0.01$, (i.e. energy part is contributing very little), and take any value of c, for example here we have taken $c = 2; c = 15; c = -20$. The phase plots of Ueda oscillator for these parameter values are given in Figure 1:



Figure(1): Phase plots of Ueda oscillator for
$a = 1; b = 0.06; \omega = 1; \epsilon = 0.01$ and $c = 2, 15, -20$

Above phase plots of Figure (1) show the regular motion of the oscillator for initial and later orbits, in general for any orbits. Therefore if the energy tends to zero in the system, the motion will tend to be regular and not chaotic. Hence, the energy part plays a very significant role in the immergence of chaos.

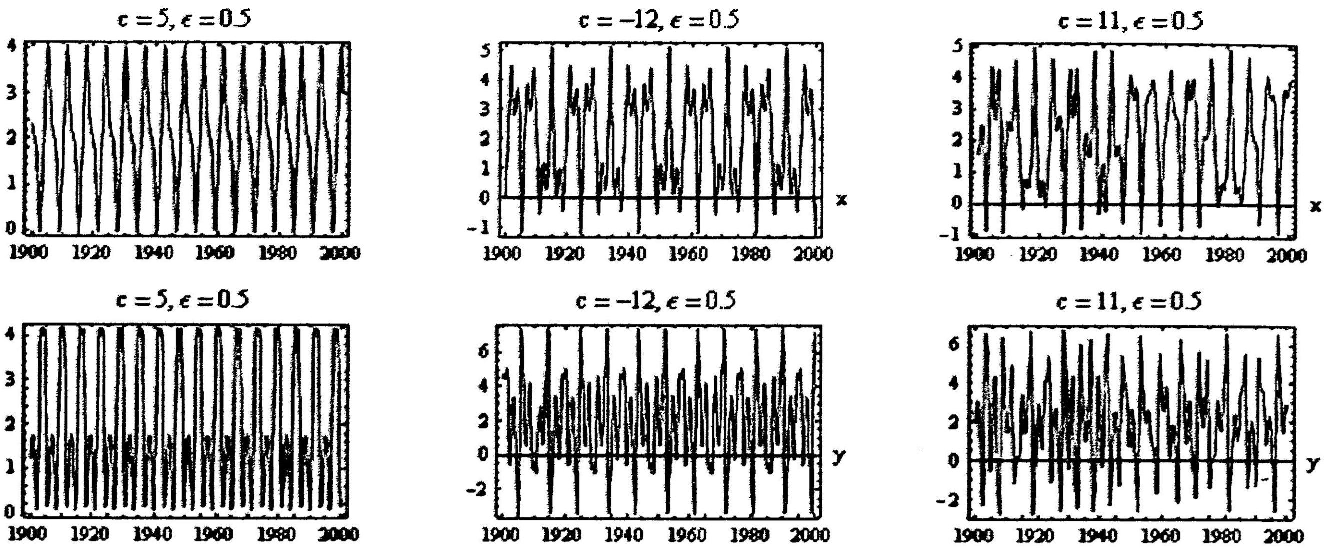
Now, we take an another value of $\epsilon = 0.5$, i.e. energy part is contributing significantly but as a variable and not as a constant. Then, we observe that the motion of the oscillator is not completely regular for any value of c. For example, if we take $c = 5; c = -12; c = 11$, the time series graphs of x and y for the time duration $t = 0$ to $t = 100$ are given as in Figure (2):



Figure(2): Time series for $t = 0$ to $t = 100$ of x and y of Ueda oscillator for
$a = 1; b = 0.06; \omega = 1; \epsilon = 0.5$ and $c = 5, -12, 11$

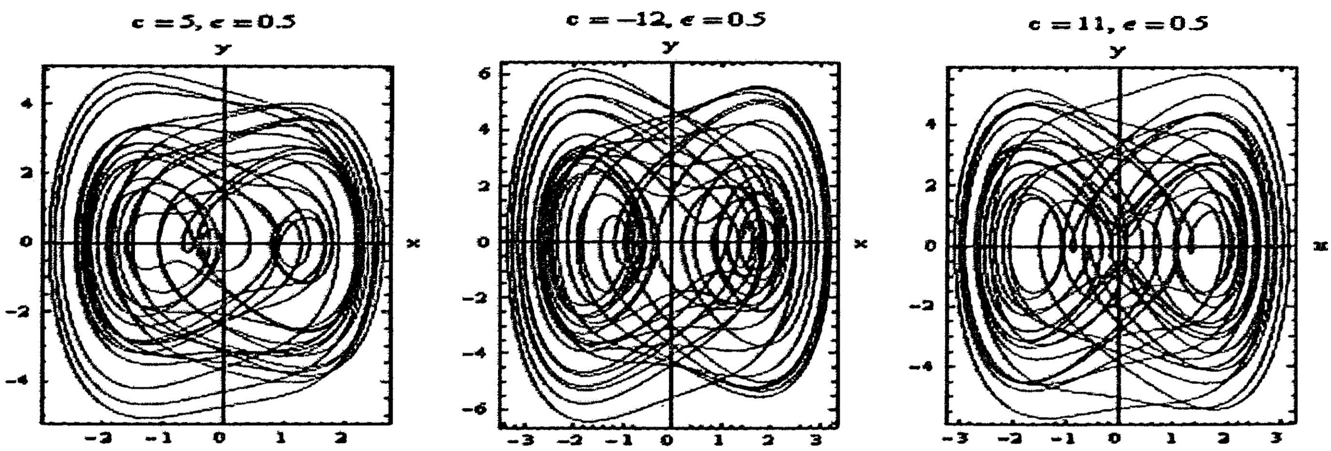
The time series graphs shown in Figure (2) display chaotic motion of the system for all the three values of c for the duration $t = 0$ to $t = 100$. Above three values of c are taken randomly for giving analytic justification of our result. In general, one would observe, for any value of c and other fixed set of parameter values, the system behaves chaotically for some certain range of time t.

However, when we increase the time duration e.g. $t = 1900$ to $t = 2000$, the behaviour of time series, as given in Figure (3), shows the regular motion for $c = 5$ and $c = -12$, but chaos for $c = 11$.



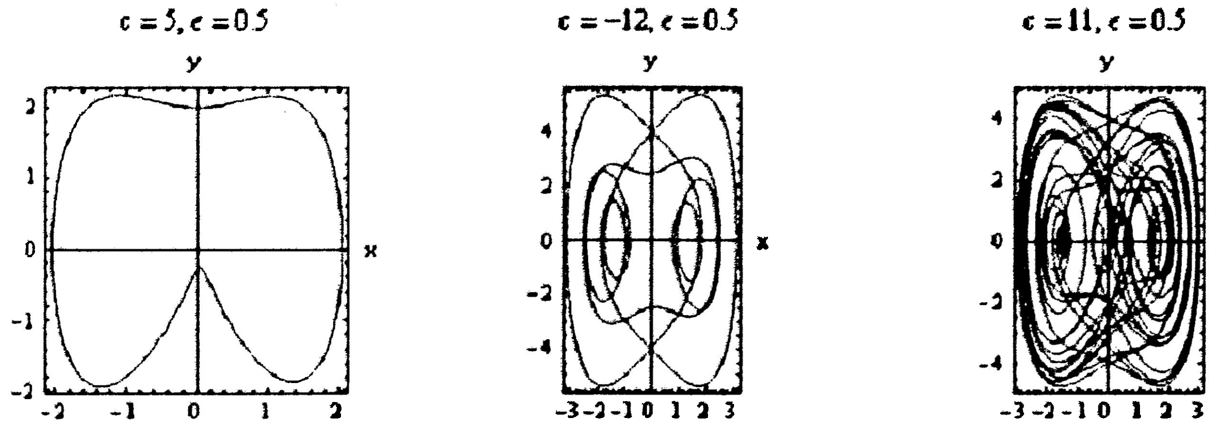
Figure(3): Time series for $t = 1900$ to $t = 2000$ of x and y of Ueda oscillator for $a = 1; b = 0.06; \omega = 1; \epsilon = 0.5$ and $c = 5, -12, 11$

We have also observed the phase plots of the oscillator between $t = 0$ to $t = 100$ and the parameter values as $a = 1; b = 0.06; \omega = 1; \epsilon = 0.5$ and $c = 5, -12, 11$, which display chaos as shown in Figure(4) below.



Figure(4): Phase plots for $t = 0$ to $t = 100$ of Ueda oscillator for $a = 1; b = 0.06; \omega = 1; \epsilon = 0.5$ and $c = 5, -12, 11$

If we draw the phase plots for exactly the same parameter values as in case of Figure(4) but for time between $t = 1900$ to $t = 2000$, then the motion reflected regular for the first two parameter values i.e. $c = 5, -12$ and chaotic for $c = 11$ as shown in Figure(5).

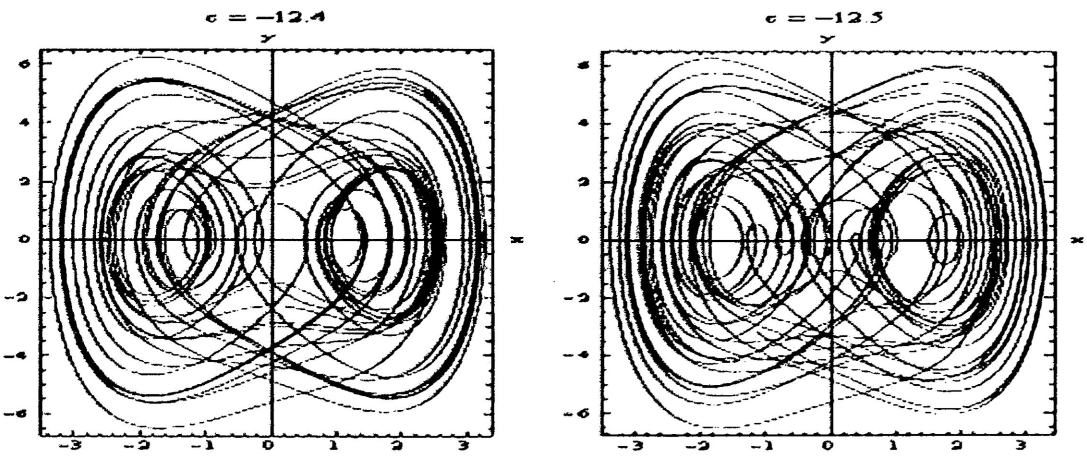


Figure(5): Phase plots for $t = 1900$ to $t = 2000$ of Ueda oscillator for $a = 1; b = 0.06; \omega = 1; \epsilon = 0.5$ and $c = 5, -12, 11$

From these graphs we have observed that the Ueda oscillator displays transient chaos for $c = 5$ as well as $c = -12$ and other fix parameter values. But for $c = 11$, it displays chaotic motion. Thus we conclude that the evolutionary behaviour of Ueda oscillator may be chaotic or transient chaotic depending on the parameter value of the driving force. Also, the energy variability has an important role in displaying transient chaos and chaos in Ueda oscillator.

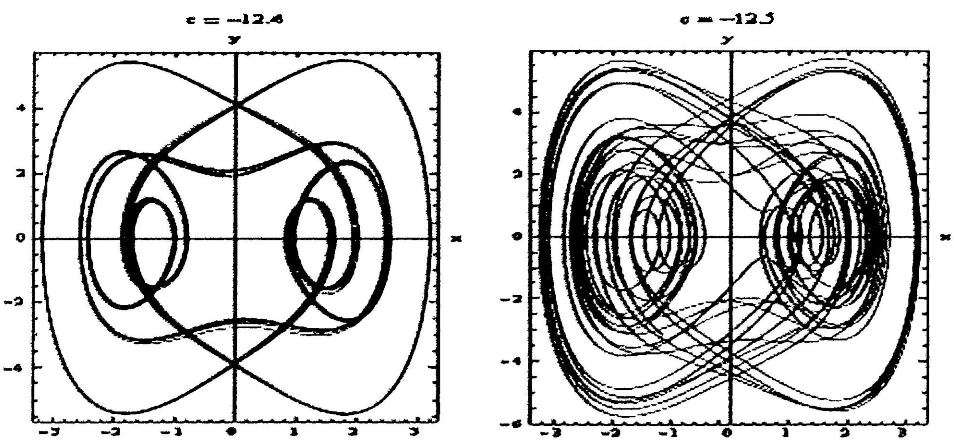
2.3 Sensitybity of Driven Force and Transient Chaos:

It has been observed that the driving force of Ueda oscillator is very sensitive for displaying the evolutionary behaviour. A minute change in the parameter c transforms the motion from chaos to transient chaos which in long term displays regularity. To show this, let us take the parameter values as $a = 1; b = 0.06; \omega = 1; \epsilon = 0.5$; and two values of c as $c = -12.4$ as well as $c = -12.5$. These two values of c are very close to each other but they show a great change in the behaviour of motion of the oscillator. In Figure (6), we have shown the phase plots of Ueda oscillator for $t = 0$ to $t = 100$ has been shown for $c = -12.4$ and $c = -12.5$. Both plots are showing chaotic structure of system but with a change in orbital structure.



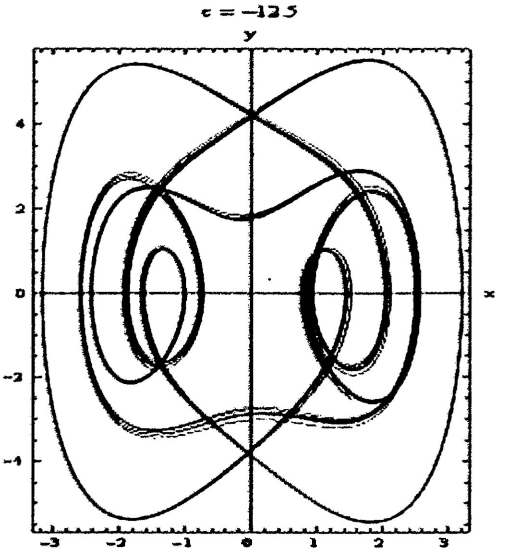
Figure(6): Phase plots for $t = 0$ to $t = 100$ of Ueda oscillator for
$a = 1; b = 0.06; \omega = 1; \epsilon = 0.5$ and $c = -12.4$ and$c = -12.5$

For the exactly same parametric values, if we take the phase plots for $t = 100$ to $t = 200$ as given in Figure(7), we can see fully regular behaviour of the system for $c = -12.4$ but chaotic for $c = -12.5$. Thus, from Figure(6) and Figure(7) we may conclude that the system is transient chaotic for $c = -12.4$. This shows the sensitivity nature of parameter c.



Figure(7): Phase plots for $t = 100$ to $t = 200$ of Ueda oscillator for
$a = 1; b = 0.06; \omega = 1; \epsilon = 0.5; c = -12.4$ and $c = -12.5$

Continuing the above process of calculation, for the parameter value $c = -12.5$, if we draw phase plot for $t = 250$ to $t = 350$, we see again a regular behaviour of the system as shown in Figure(8). Therefore this parameter value, $c = -12.5$ again displays the transient chaos.

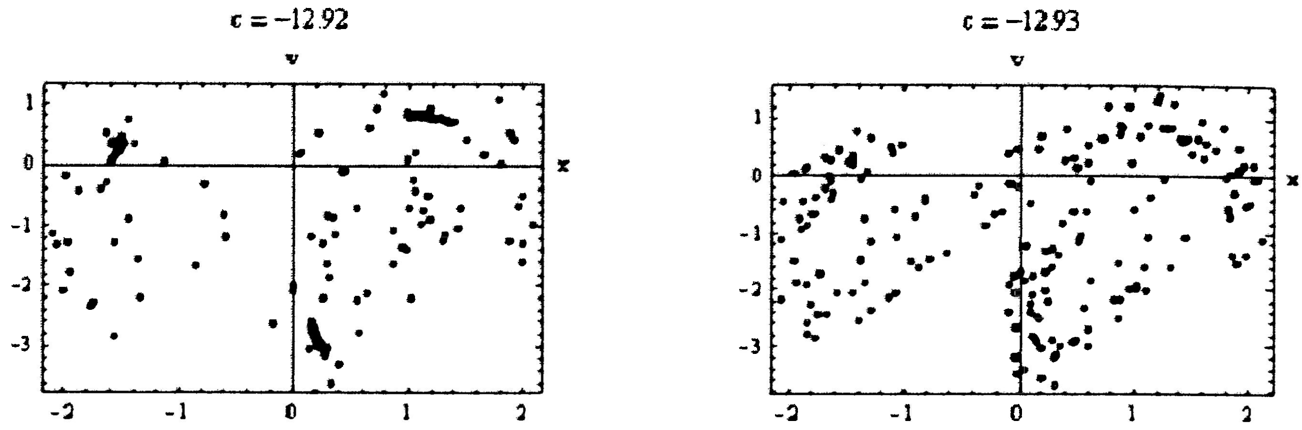


Figure(8): Phase plots for $t = 250$ to $t = 350$ of Ueda oscillator for $a = 1; b = 0.06; \omega = 1; \epsilon = 0.5$ and $c - 12.5$

Figures(6, 7, 8), clearly show the sensitivity of parameter c and immergence of transient chaos in the Ueda oscillator. The regularity in the first case appears at $t = 70$ onwards (approx.) and that in the second case appears at $t = 250$ onwards(approx.).

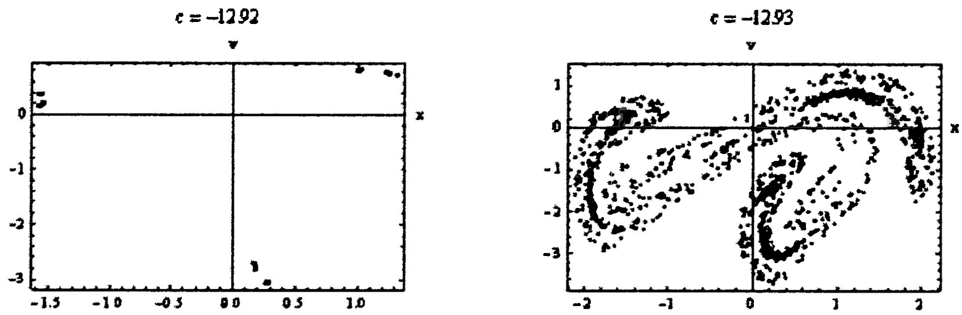
2.4 Poincare Section for Chaos and Transient Chaos:

The chaos is best exhibited in a Poincare section, in which x and the angular velocity $\frac{dx}{dt}$ are plotted at a constant phase of the drive where ωt is an integral multiple of 2π. Earlier we have seen the transient property of the system for two nearby parameter values of driving force c displaying with the sensitivity in the orbits through phase plots. Here, we deal again with two very near parameter values $c = -12.92$ and $c = -12.93$. For the first we get the transient chaotic system and for the second we get fully chaotic system. For this, let us take in the Ueda oscillator, parameter values $a = 1; b = 0.06; \omega = 1; \epsilon = 0.5$; together with values $c = -12.92$ and $c = -12.93$. In Figure(9), we have shown the Poincaré surface section for these cases for the orbits between $t = 200$ to $t = 300$.



Figure(9): Poincare Surface Section for $t = 200$ to $t = 300$ of Ueda oscillator for $a = 1; b = 0.06; \omega = 1; \epsilon = 0.5; c = -12.92$ and $c - 12.93$

Figure(9) shows the chaotic motion for both the cases. However, as given in Figure(10), the Poincaré surface section for the same parameter values but between, $t = 1900$ to $t = 2000$. From this figure, it is clear that the system is regular for $c = -12.92$ and chaotic for $c = -12.93$. Figure(9) and Figure(10) jointly show the transient chaotic motion of the system for $c = -12.92$ and fully chaotic motion for $c = -12.93$.



Figure(10): Poincare Surface Section for $t = 1900$ to $t = 2000$ of Ueda oscillator for $a = 1; b = 0.06; \omega = 1; \epsilon = 0.5; c = -12.92$ and $c - 12.93$

Since, these two parameters are very close to each other, we observed very significant change in the Poincaré surface sections of the system. For a very little change in the parameter c transforms a fully chaotic system to transient chaotic system. This result for Ueda oscillator provides us a method of controlling chaos by using the sensitivity characteristic of the driven parameter.

3 Conclusion:

This paper gives a specific emphasise over variable energy and its relation with chaos and transient chaos emerging in the Ueda oscillator. Since the natural system in this world are mostly nonlinear and enormous in number, the results obtained for Ueda oscillator may also be observe in some other oscillator. It is a small effort in the direction of research dealing with variability of the energy. Control of chaos with a very small change in a particular parameter attracts our interest of research towards chaos control. Since the behaviour of nonlinear systems is unpredictable, these results may not be generalized to all other nonlinear oscillators. But certainly this type of investigation to many other systems may definitely produce very interesting results. In nature one can find numerous systems evolving through some driven force, which are associated to our day to day life and this type of study may give interesting revelation.

As we have seen from graphical representation and numerical analysis of Ueda oscillator with different strength of driven force, the system is very sensitive to energy and driving force. This kind of discussion for other useful nonlinear systems will be presented in our later papers.

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