生駒経済論叢　第７巻第１号　2009年７月

A Theory of Logistic Curve
――To Explain the Transition of Japanese Economy after World War Ⅱ――

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Abstract The Japanese Economy after World War Ⅱ has greatly changed. It is a general understood that the changes or the transition of Japanese Economy could be explained not with linear dynamics but with nonlinear dynamics. We have a theoretical hypothesis that the transition of the Japanese Economy after World War Ⅱ would be rationally explained using a logistic curve (that is, an S curve). Accordingly, it is our purpose to clarify this in the next three issues in this paper. The first issue will formulate a logistic curve mathematically. Second issue will prove the chaos arising from the final stage of the changes to the logistic curve. Third issue will clarify whether the transition of Japanese Economy after World War Ⅱ could be explained rationally using a logistic curve or not.

Key words real logistic equation, logistic curve, logistic mapping equation, mapping parameter, Chaos, Japanese Economy after World War Ⅱ

April 29, 2009 accepted

Ⅰ. Introduction

Generally speaking, it is a common understanding that a typical changes of any systems which consist of Birth→the period of Growth→the period of maturity →Chaos (→New Birth) could be explained with nonlinear dynamics (for instance, with logistic curve).

It is needless to say that any changes of Japanese Economy after World War Ⅱ are no exception.

We have an idea that nonlinear logistic curve is quite effective to explain rationally the changes of Japanese Economy after World War Ⅱ.

Accordingly, it is our purposes in this paper to explain theoretically next three issues.

First, What is the logistic curve? How can we formulate it mathematically? (It would be discussed in Chapter Ⅱ.)

Second, What is a meanings that the theory of logistic curve could explain the chaos phenomena? (It would be discussed in Chapter Ⅲ.)

Finally, How could we explain the transition of Japanese Economy after World War Ⅱ with logistic curve analogically? (It would be discussed in Chapter Ⅳ.)

Ⅱ. Real Logistic Equation

(1) Derivation of Real Logistic Equation

Considering N as an economic variable, we assume the logistic curve(S curve) in relation to the real time t as follows.

$$\frac{dN}{dt} = aN + bN^2, \qquad \text{where } a>0, \quad b<0 \qquad ①$$

When t tends to $+\infty$, N attains a maximum $\overline{K}$ which satisfies the following equation.

$$\left.\frac{dN}{dt}\right|_{N=\bar{K}} = a\bar{K} + b\bar{K}^2 = 0$$

$$b = -\frac{a}{\bar{K}} \qquad ②$$

Putting ② into ①, we have a real logistic equation ③

$$\frac{dN}{dt} = aN - \frac{a}{\bar{\bar{K}}}N^2 = a\left(1 - \frac{N}{\bar{\bar{K}}}\right)N \qquad ③$$

Multiplying the equation ③ by $\frac{1}{\bar{\bar{K}}}$ on both sides, we get

$$\frac{d}{dt}\left(\frac{N}{\bar{\bar{K}}}\right) = a\left(1 - \frac{N}{\bar{\bar{K}}}\right)\frac{N}{\bar{\bar{K}}} \qquad ④$$

Here, we replace $\frac{N}{\bar{\bar{K}}}$ with N again. Then, we have the following equation.

$$\frac{dN}{dt} = a(1-N)N, \qquad \text{where } a > 0 \qquad (1)$$

In this paper, we call this equation (1) as the real logistic equation.

(2) Behavior of Solutions of Real Logistic Equation

If the initial value N_0 is different from 0 and 1, we can concretely solve the equation(1):

The solution of the equation (1) with the initial point N_0 is the equation (2).

$$N(t) = \frac{\frac{N_0}{1-N_0}}{e^{-at} + \frac{N_0}{1-N_0}} \qquad (2)$$

Especially, if N_0 is 1, we have $N_{(t)} \equiv 1$ and if N_0 is 0, we have $N_{(t)} \equiv 0$.

We can classify the solution (2) the following five cases:

Case 1 : $0 < N_0 = N_2 < 1$

$$N(t) = \frac{\frac{N_2}{1-N_2}}{e^{-at} + \frac{N_2}{1-N_2}} \qquad (2)^{[1]}-①$$

Case 2： $N_0 = N_3 > 1$

$$N(t) = \frac{\dfrac{N_3}{N_3 - 1}}{\dfrac{N_3}{N_3 - 1} - e^{-at}} = \frac{\dfrac{N_3}{1 - N_3}}{e^{-at} + \dfrac{N_3}{1 - N_3}} = \frac{\bar{C}}{e^{-at} + \bar{C}} \qquad (2)^{[2]} - ②$$

where $\bar{C} = \dfrac{N_3}{1 - N_3} < 0$

Case 3： $N_0 = N_1 < 0$

$$N(t) = \frac{\dfrac{N_1}{1 - N_1}}{e^{-at} - \dfrac{-N_1}{1 - N_1}} = \frac{\dfrac{N_1}{1 - N_1}}{e^{-at} + \dfrac{N_1}{1 - N_1}} = \frac{\bar{C}}{e^{-at} + \bar{C}} \qquad (2)^{[3]} - ③$$

where $\bar{C} = \dfrac{N_1}{1 - N_1} < 0$

Case 4： $N_0 = 0$

$$N_{(t)} \equiv 0 \qquad (2) - ④$$

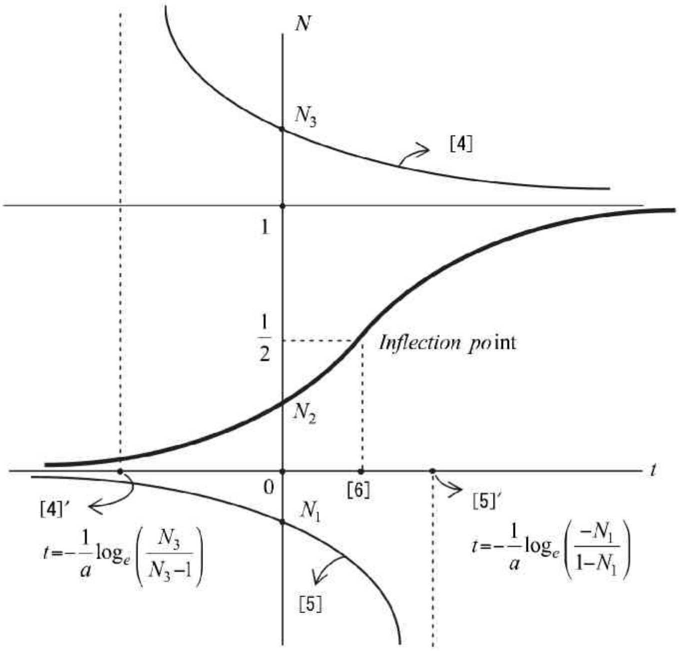

Fig. 1

Case 5 : $N_0 = 1$

$$N_{(t)} \equiv 1 \qquad (2)-⑤$$

In [Figure 1] , we can draw the graphs of Case 1–Case 5 mentioned above. From now on, the logistic curve of Case 1, that is a standard type, would be used in the following discussion.

Ⅲ. Derivation of Logistic Mapping and its Natures

(1) Derivation of Logistic Mapping Equation

$$\frac{dN}{dt} = a\left(1 - \frac{N}{\bar{K}}\right)N \qquad ③$$

Let's rewrite the differential equation ③ to the difference equation as follows.

$$\left(\frac{dN}{dt} = \right)\frac{N(t+\Delta t) - N(t)}{\Delta t} = a\left(1 - \frac{N(t)}{\bar{K}}\right)N(t)$$

$$N(t+\Delta t) = \Delta taN(t) - \frac{\Delta taN^2(t)}{\bar{K}} + N(t)$$

$$= \left(\Delta ta - \frac{\Delta taN(t)}{\bar{K}} + 1\right)N(t)$$

Here, we replace $\Delta ta + 1$ to α

$$= \left(\alpha - \frac{\Delta taN(t)\alpha}{\bar{K}\alpha}\right)N(t)$$

Multiplying $\frac{\Delta ta}{\bar{K}\alpha}$ both sides,

$$\frac{N(t+\Delta t)\cdot \Delta ta}{\bar{K}\alpha} = \left(\alpha - \frac{\Delta taN(t)\alpha}{\bar{K}\alpha}\right)\cdot \frac{\Delta taN(t)}{\bar{K}\alpha}$$

Here, we replace $\frac{N(t+\Delta t)\cdot \Delta ta}{\bar{K}\alpha} = x_{n+1}$, and $\frac{N(t)\cdot \Delta ta}{\bar{K}\alpha} = x_n$

We get the next difference equation, that is, the logistic mapping equation.

$$x_{n+1} = \alpha(1 - x_n)x_n , \quad \text{where} \quad 0 < \alpha \leq 4 \qquad (3)$$

(2) Effects of the Mapping Parameter α on the Shape of Logistic Curves

$$x_{n+1} = -\alpha x_n^2 + \alpha x_n = -\alpha\left(x_n - \frac{1}{2}\right)^2 - \frac{\alpha^2}{4\cdot(-\alpha)} \qquad (3)'$$

(3)′ equation is a quadratic equation as to concave to the above in relation to x_n. The maximum points of the equation (3)′ is $\left(\frac{1}{2},\frac{\alpha}{4}\right)$.

Taking a mapping parameter α to $0<\alpha\le 4$, and drawing a graph of equation (3) within a regular square, that is, $(x_n, x_{n+1}) = (1,1)$, we can get the change of logistic mappings in Fig. 2.

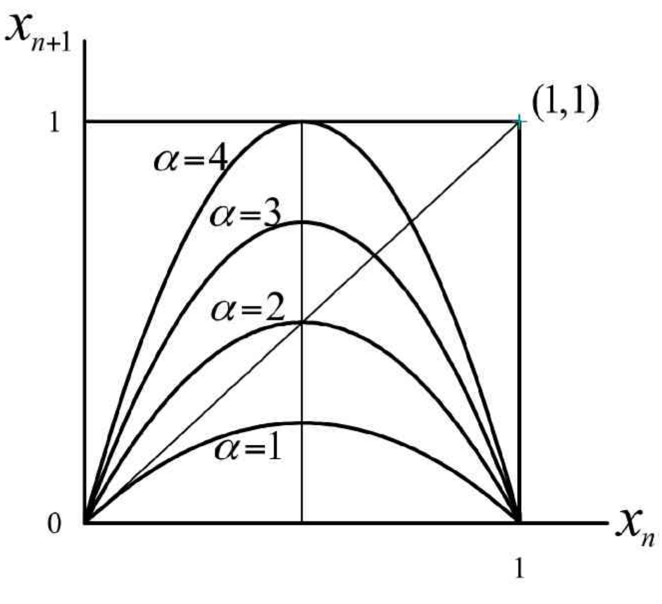

Fig. 2

Taking the change in the shape of logistic curve corresponding to the change of α that is increasing from 1.5 to 3.6, we can get the next 10 figures. It is very interesting phenomena that a chaos would appear when mapping parameter α would increase to 3.6 above.

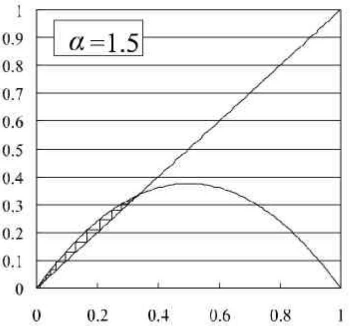

Fig. 3-a

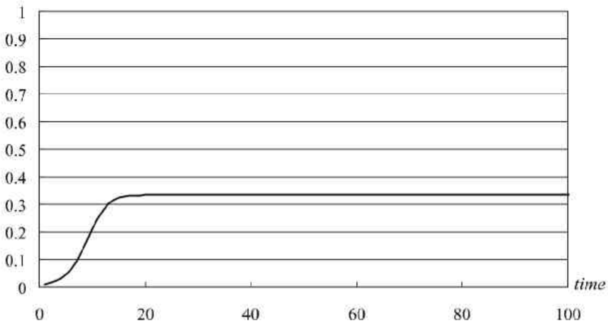

Fig. 3-b

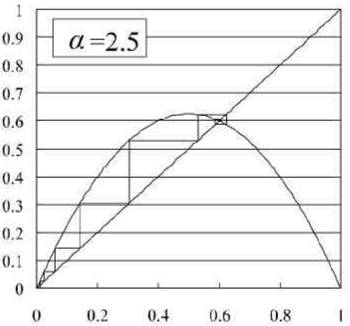

Fig. 4-a

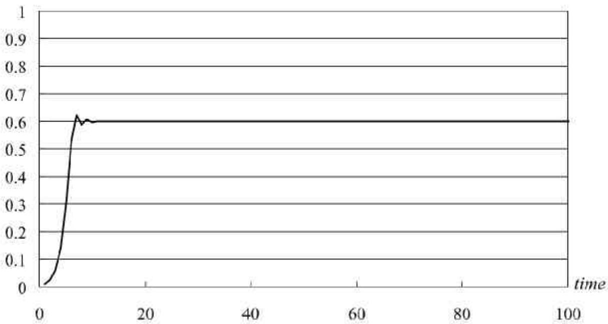

Fig. 4-b

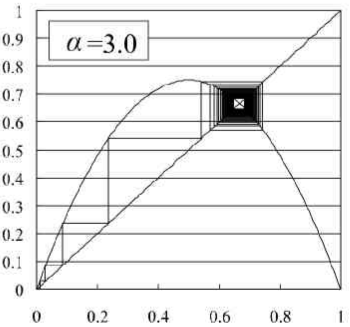

Fig. 5-a

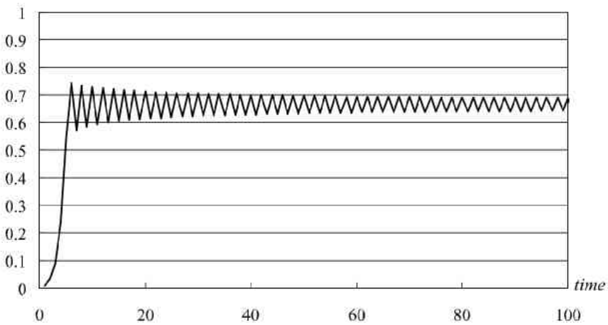

Fig. 5-b

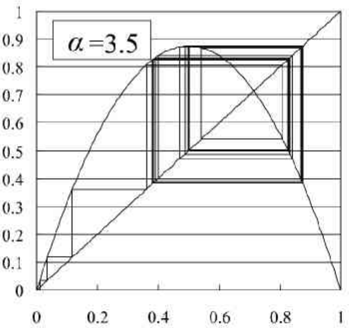

Fig. 6-a

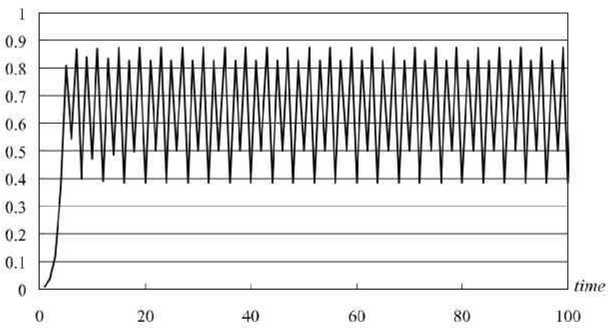

Fig. 6-b

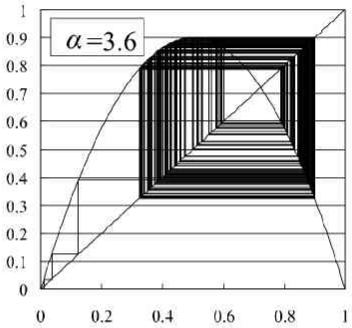

Fig. 7-a

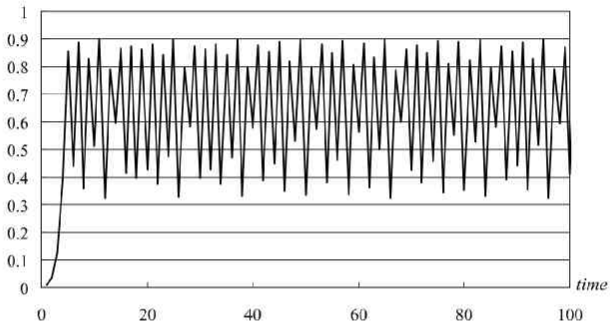

Fig. 7-b

Ⅳ. Concluding Remarks

(1) Specification of the Shape of Logistic Curve for the Actual Japanese Economy

Nothing to say that in the actual economy, the mapping parameter α has changed with the passage of time. Therefore, we have to remake the logistic curve in Fig. 3-b—Fig. 7-b so as to reflect the change of actual mapping parameter α. Consequently, we can get the Fig. 8 in three dimensions as below.

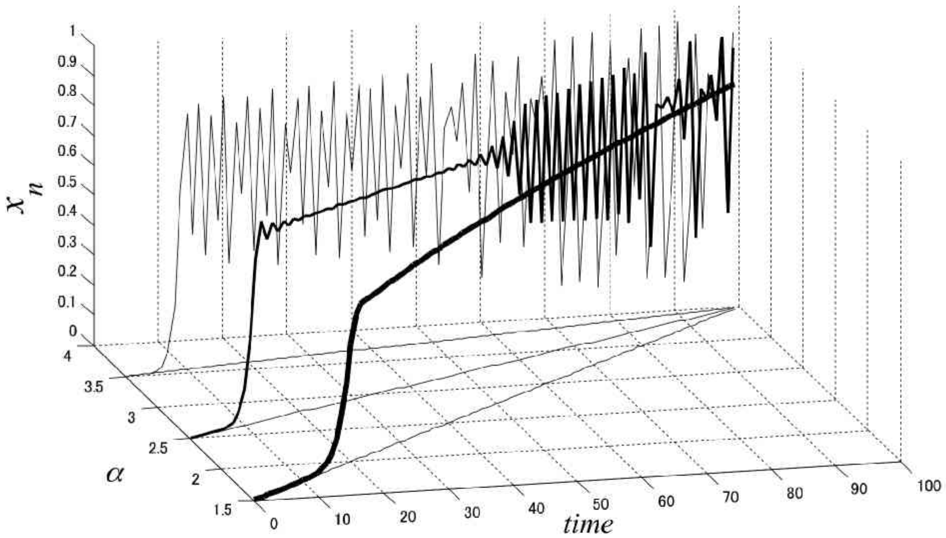

Fig. 8

We believe that it is quite useful for the logistic curve including chaos in Fig. 8 to explain the actual behavior of the Japanese Economy.

(2) Analogical Explanation with Logistic Curve for the Actual Japanese Economy

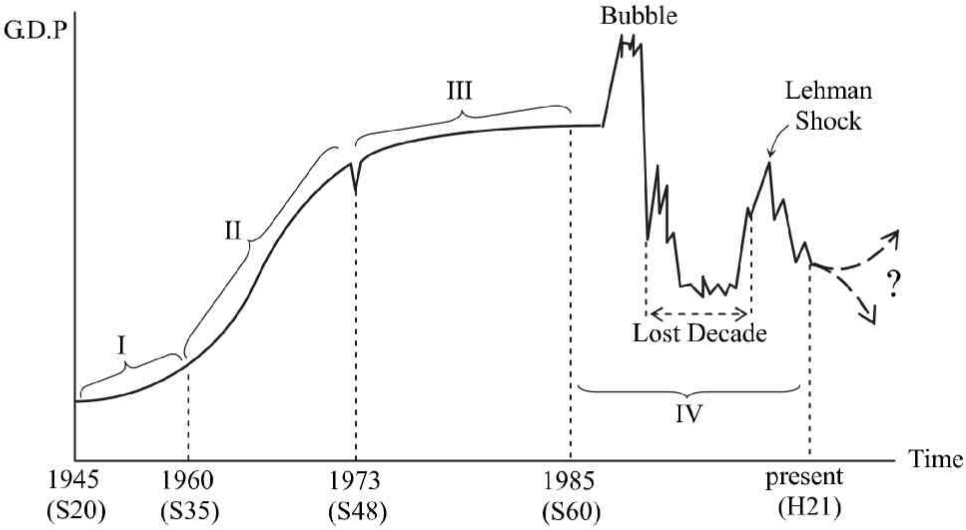

Fig. 9

【Period I (1945-1960)】

- Post War any Constructions based on Democracy
- Basic and Institutional Repairments
- Ages which Politics lead Economy
- People lived in Poverty
- Ages of ideology that is, Capitalism or Socialism ?

【Period II (1960-1973)】

- Ages of High Economic Growth under the Leadership of Private Economy
- Ages of Keynes & Keynesian Economics
- Combination Multiplier Effect of Investment with Acceleration Principle
- Ages of Economy

【Period Ⅲ (1973-1985)】

- Oil shocks extending over two times
- Ages of Stagflation (that is, Coexistence of High Inflation and Deep Depression)
- Ages of Zero Growth Rate
- International Policy Coordination (that is, SUMMIT)
- Ages which Politics intervene in Economy

【Period Ⅳ (1985～up to now)】

- In this Period Ⅳ, vertical axis is not G.D.P but the Rate of Change of it.
- In a word, Ages of Chaos
- Plaza Accord (9.1985) ⇒ Approval of the Quick Appreciation of Yen ⇒ Domestic demanded Growth ⇒ Monetary Relaxation ⇒ Existence of the Excess Liquidity ⇒ Bubble based on the linkage of Stocks and Land ⇒ Bubble Collapse ⇒ Lost Decade (in Correctly, Lost Fifteen or Twenty Years) ⇒ Worldwide Financial Crisis caused by Lehman Brothers Shock
- Politics and Economy in the world still exist in the midst of Chaos.

Mathematical footnotes

[1] Let the equation (1) be changed by equation (1)′.

$$\frac{dN}{dt} = a(1-N)N, \quad \text{where} \quad a>0,\ N \neq 0,1 \tag{1}$$

$$\frac{dN}{(1-N)N} = adt$$

$$\frac{dN}{N} + \frac{dN}{1-N} = adt \tag{1′}$$

Case 1 : $0 < N_0 = N_2 < 1$

$$\int_{N_2}^{N}\left(\frac{dN}{N}-\frac{d\left(1-N\right)}{1-N}\right)=\int_0^t adt$$

$$\log_e N-\log_e N_2-\left(\log_e(1-N)-\log_e(1-N_2)\right)=at$$

$$\log_e\frac{N}{N_2}-\log_e\frac{1-N}{1-N_2}=at$$

$$\log_e\left(\frac{N}{N_2}\times\frac{1-N_2}{1-N}\right)=at$$

$$\frac{N}{1-N}=\frac{N_2}{1-N_2}\cdot e^{at}$$

Solving the above equation for N, we get

$$N(t)=\frac{\dfrac{N_2}{1-N_2}}{e^{-at}+\dfrac{N_2}{1-N_2}} \qquad (2)-①$$

[2]

Case 2 : $1<N_0=N_3$

$$\int_{N_3}^{N}\left(\frac{dN}{N}-\frac{d(N-1)}{N-1}\right)=\int_0^t a\cdot dt$$

$$\log_e N-\log_e N_3-\left(\log_e(N-1)-\log_e(N_3-1)\right)=at$$

$$\frac{N}{N_3}\Big/\frac{N-1}{N_3-1}=e^{at}$$

Solving the above equation for N, we get

$$N(t)=\frac{\dfrac{N_3}{N_3-1}}{\dfrac{N_3}{N_3-1}-e^{-at}}=\frac{\dfrac{N_3}{1-N_3}}{e^{-at}+\dfrac{N_3}{1-N_3}}=\frac{\overline{C}}{e^{-at}+\overline{C}} \qquad (2)-②$$

[3]

Case 3 : $N_0=N_1<0$

$$\int_{N_1}^{N}\left(\frac{d(-N)}{-N}-\frac{d(1-N)}{1-N}\right)=\int_0^t a\cdot dt$$

$$\log_e(-N)-\log_e(-N_1)-\left(\log_e(1-N)-\log_e(1-N_1)\right)=at$$

$$\log_e\left(\frac{-N}{-N_1}\right)-\log_e\left(\frac{1-N}{1-N_1}\right)=at$$

$$\frac{N}{1-N}\times\frac{1-N_1}{N_1}=e^{at}$$

$$N(t)=\frac{\dfrac{N_1}{1-N_1}}{e^{-at}+\dfrac{N_1}{1-N_1}}=\frac{\bar{C}}{e^{-at}+\bar{C}} \qquad (2)-③$$

[4] The derivation of the curve of initial value N_3 in Figure 1.

$$N(t)=\frac{\dfrac{N_0}{1-N_0}}{e^{-at}+\dfrac{N_0}{1-N_0}}, \quad \text{where} \quad a>0 \qquad (2)$$

If we take N_0 in equation (2) to $\bar{N}_3>1$, we have the equation (2)－②.

$$N(t)=\frac{C}{e^{-at}+C}, \quad \text{where} \quad \frac{\bar{N}_3}{1-\bar{N}_3}=C<0 \qquad (2)-②$$

In order to confirm the property of the equation (2)−②, we have to examine the sign the first / second order differential with respect to real time t of equation (2)－②.

(i) $$\frac{dN}{dt}=\frac{-C\cdot e^{-at}\cdot(-a)}{(e^{-at}+C)^2}=\frac{C\cdot a\cdot e^{-at}}{(e^{-at}+C)^2}<0$$

Therefore, the function $N(t)$ is monotone decreasing.

(ii) $$\frac{d^2N}{dt^2}=\frac{Ca\cdot e^{-at}\cdot(-a)\cdot(e^{-at}+C)^2-2(e^{-at}+C)\cdot e^{-at}\cdot(-a)\cdot Ca\cdot e^{-at}}{(e^{-at}+C)^4}$$

$$=\frac{-C\cdot a^2\cdot e^{-at}\cdot(C-e^{-at})}{(e^{-at}+C)^3}$$

Since $\bar{C}$ is negative and a is positive, the numerator of the above equation is always negative.

In order to confirm the sign of denominator, put $e^{-at}+C$ to $h(t)$. We here draw the graph of $h(t)$.

$$h(t)=e^{-at}+C=0$$
$$e^{-at}=-C$$

Taking a natural logarithm on both sides. We have

$$t=-\frac{1}{a}\log(-C)$$

Then, the sign of denominator is always negative, if we take t larger than $-\frac{1}{a}\log(-C)$.

Under the above investigation, the sign of second order differential is positive.

After all, we can draw the graph in Figure 1.

[4]′ We want to ask the range of the time for $N(t)>1$. Watch the first term of equation (2)−② in Mathematical footnotes [2]. Since e^{-at} is positive ($a>0$), we can find that $N(t)$ is always greater than 1 as long as the denominator is positive.

$$\frac{N_3}{N_3-1}-e^{-at}>0$$

$$e^{-at}<\frac{N_3}{N_3-1}$$

Taking a natural logarithm

$$-at<\log_e\left(\frac{N_3}{N_3-1}\right) \qquad a>0$$

$$t>-\frac{1}{a}\log_e\left(\frac{N_3}{N_3-1}\right)$$

[5] The derivation of the curve of initial value N_1 in Figure 1.

$$N(t)=\frac{\frac{N_0}{1-N_0}}{e^{-at}+\frac{N_0}{1-N_0}}, \text{ where } a>0 \qquad (2)$$

If we take N_0 in equation (2) to $N_1 < 0$, we have the equation (2)－③

$$N(t)=\frac{C}{e^{-at}+C}, \quad \text{where} \quad \frac{\bar{N}_1}{1-\bar{N}_1}=C<0 \tag{2)'}$$

In order to confirm the shape of the equation (2)-③, we have to examine the sign of the first / second order differential with respect to real time t of equation (2)－③.

(i) $$\frac{dN}{dt}=\frac{C\cdot a\cdot e^{-at}}{(e^{-at}+C)^2}<0$$

Accordingly, the function $N(t)$ is monotone decreasing.

(ii) $$\frac{d^2N}{dt^2}=\frac{-C\cdot a^2\cdot e^{-at}\cdot(C-e^{-at})}{(e^{-at}+C)^3}$$

Taking account of $\bar{C}<0$ and $a>0$, the numerator of the above equation is always negative.

In order to confirm the sign of denominator, we will draw the graph of $h(t)=e^{-at}+C$.

If real time $t\,(<0)$ is smaller than $-\frac{1}{a}\log_e\left(\frac{-N_1}{1-N_1}\right)$, $h(t)$ is always positive. That is, as we can understand that the numerator is negative and the denominator is positive of the second order differential equation, equation (2)-③ will be shown as a curve in Figure 1.

[5]′

$$h(t)=e^{-at}+C=0$$

$$e^{-at}=-C, \quad \bar{C}=\frac{N_1}{1-N_1}<0$$

Taking a natural logarithm on both sides,

$$t=-\frac{1}{a}\log_e-C=-\frac{1}{a}\log_e\left(\frac{-N_1}{1-N_1}\right)>0$$

[6] Take second differential of equation ① with respect to real time t.

$$\frac{d^2N}{dt^2} = a \cdot \frac{dN}{dt} + 2bN \cdot \frac{dN}{dt} = (a + 2bN) \cdot \frac{dN}{dt}$$

At inflection point, $\frac{d^2N}{dt^2} = 0$. And so,

$$(a + 2bN) \cdot \frac{dN}{dt} = (a + 2bN)(a + bN)N = 0$$

Excluding $N = 0$, $-\frac{a}{b}(=\bar{K} = 1)$, $N = -\frac{a}{2b} = \frac{1}{2}$.

Accordingly, the time which satisfy the equation (2)-② $=$ $\frac{1}{2}$ is the one we are seeking for.

$$N(t) = \frac{\frac{N_2}{1 - N_2}}{e^{-at} + \frac{N_2}{1 - N_2}} = \frac{1}{2}$$

$$\therefore t = -\frac{1}{a} \log_e \left(\frac{N_2}{1 - N_2} \right)$$

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