Balanced \((C_5, C_{14})-2t\)-Foil Decomposition Algorithm of Complete Graphs

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In graph theory, the decomposition problem of graphs is a very important topic. Various types of decompositions of many graphs can be seen in the literature of graph theory. This paper gives a balanced \((C_5, C_{14})-2t\)-foil decomposition of the complete graph \(K_n\).

2. Balanced \((C_5, C_{14})-2t\)-foil decomposition of \(K_n\)

**Theorem.** \(K_n\) has a balanced \((C_5, C_{14})-2t\)-foil decomposition if and only if \(n \equiv 1 \pmod{38t}\).

**Proof.** (Necessity) Suppose that \(K_n\) has a balanced \((C_5, C_{14})-2t\)-foil decomposition. Let \(b\) be the number of \((C_5, C_{14})-2t\)-foils and \(r\) be the replication number. Then \(b = n(n - 1)/38t\) and \(r = (17t + 1)(n - 1)/38t\). Among \(r\) \((C_5, C_{14})-2t\)-foils having a vertex \(v\) of \(K_n\), let \(r_1\) and \(r_2\) be the numbers of \((C_5, C_{14})-2t\)-foils in which \(v\) is the center and \(v\) is not the center, respectively. Then \(r_1 + r_2 = r\). Counting the number of vertices adjacent to \(v\), \(4r_1 + 2r_2 = n - 1\). From these relations, \(r_1 = (n - 1)/38t\) and \(r_2 = 17(n - 1)/38\). Therefore, \(n \equiv 1 \pmod{38}\) is necessary.

(Sufficiency) Put \(n = 38st + 1\) and \(T = st\). Then \(n = 38T + 1\).

Construct a \((C_5, C_{14})-2T\)-foil as follows:

\[
\{(38T + 1, 1, 14T + 2, 34T + 3, 16T + 1), (38T + 1, T + 1, 6T + 2, 11T + 2, 21T + 3, 29T + 3, 6T + 3, 18T + 3, 8T + 3, 5T + 2, 30T + 3, 24T + 2, 21T + 2, 13T + 1)\} \cup \\
\{(38T + 1, 2, 14T + 4, 34T + 6, 16T + 2), (38T + 1, T + 2, 6T + 4, 11T + 3, 21T + 5, 29T + 4, 6T + 5, 18T + 4, 8T + 5, 5T + 3, 30T + 5, 24T + 2, 21T + 4, 13T + 2)\} \cup \\
\{(38T + 1, 3, 14T + 6, 34T + 9, 16T + 3), (38T + 1, T + 3, 6T + 6, 11T + 4, 21T + 7, 29T + 5, 6T + 7, 18T + 5, 8T + 7, 5T + 4, 30T + 7, 24T + 4, 21T + 6, 13T + 3)\} \cup \\
\ldots \cup \\
\{(38T + 1, T, 16T, 37T, 17T), (38T + 1, 2T, 8T, 12T + 1, 23T + 1, 30T + 2, 8T + 1, 19T + 2, 10T + 1, 6T + 1, 32T + 1, 25T + 1, 23T, 14T)\}.
\]

Decompose the \((C_5, C_{14})-2T\)-foil into \(s\) \((C_5, C_{14})-2t\)-foils. Then these starters comprise a balanced \((C_5, C_{14})-2t\)-foil decomposition of \(K_n\).

**Corollary.** \(K_n\) has a balanced \((C_5, C_{14})\)-bowtie decomposition if and only if \(n \equiv 1 \pmod{38}\).
Example 1. Balanced \((C_5, C_{14})\)-2-foil decomposition of \(K_{39}\).
\[
\{(39, 1, 16, 37, 17), (39, 2, 8, 13, 24, 32, 9, 21, 11, 7, 33, 26, 23, 14)\}.
\]
This starter comprises a balanced \((C_5, C_{14})\)-2-foil decomposition of \(K_{39}\).

Example 2. Balanced \((C_5, C_{14})\)-4-foil decomposition of \(K_{77}\).
\[
\{(77, 1, 30, 71, 33), (77, 3, 14, 24, 45, 61, 15, 39, 19, 12, 63, 50, 44, 27)\} \cup
\{(77, 2, 32, 74, 34), (77, 4, 16, 25, 47, 62, 17, 40, 21, 13, 65, 51, 46, 28)\}.
\]
This starter comprises a balanced \((C_5, C_{14})\)-4-foil decomposition of \(K_{77}\).

Example 3. Balanced \((C_5, C_{14})\)-6-foil decomposition of \(K_{115}\).
\[
\{(115, 1, 44, 105, 49), (115, 4, 26, 46, 87, 90, 27, 75, 35, 22, 123, 98, 86, 53)\} \cup
\{(115, 2, 46, 108, 50), (115, 5, 22, 36, 68, 91, 23, 58, 29, 18, 95, 75, 67, 41)\} \cup
\{(115, 3, 48, 111, 51), (115, 6, 24, 37, 70, 92, 25, 59, 31, 19, 97, 76, 69, 42)\}.
\]
This starter comprises a balanced \((C_5, C_{14})\)-6-foil decomposition of \(K_{115}\).

Example 4. Balanced \((C_5, C_{14})\)-8-foil decomposition of \(K_{153}\).
\[
\{(153, 1, 58, 139, 65), (153, 5, 26, 46, 87, 119, 27, 75, 35, 22, 123, 98, 86, 53)\} \cup
\{(153, 2, 60, 142, 66), (153, 6, 28, 47, 89, 120, 29, 76, 37, 23, 125, 99, 88, 54)\} \cup
\{(153, 3, 62, 145, 67), (153, 7, 30, 48, 91, 121, 31, 77, 39, 24, 127, 100, 90, 55)\} \cup
\{(153, 4, 64, 148, 68), (153, 8, 32, 49, 93, 122, 33, 78, 41, 25, 129, 101, 92, 56)\}.
\]
This starter comprises a balanced \((C_5, C_{14})\)-8-foil decomposition of \(K_{153}\).

Example 5. Balanced \((C_5, C_{14})\)-10-foil decomposition of \(K_{191}\).
\[
\{(191, 1, 72, 173, 81), (191, 6, 32, 57, 108, 148, 33, 93, 43, 27, 153, 122, 107, 66)\} \cup
\{(191, 2, 74, 176, 82), (191, 7, 34, 58, 110, 149, 35, 94, 45, 28, 155, 123, 109, 67)\} \cup
\{(191, 3, 76, 179, 83), (191, 8, 36, 59, 112, 150, 37, 95, 47, 29, 157, 124, 111, 68)\} \cup
\{(191, 4, 78, 182, 84), (191, 9, 38, 60, 114, 151, 39, 96, 49, 30, 159, 125, 113, 69)\} \cup
\{(191, 5, 80, 185, 85), (191, 10, 40, 61, 116, 152, 41, 97, 51, 31, 161, 126, 115, 70)\}.
\]
This starter comprises a balanced \((C_5, C_{14})\)-10-foil decomposition of \(K_{191}\).

Example 6. Balanced \((C_5, C_{14})\)-12-foil decomposition of \(K_{229}\).
\[
\{(229, 1, 86, 207, 97), (229, 7, 38, 129, 177, 39, 111, 51, 32, 183, 146, 128, 79)\} \cup
\{(229, 2, 88, 210, 98), (229, 8, 40, 69, 131, 178, 41, 112, 53, 33, 185, 147, 130, 80)\} \cup
\{(229, 3, 90, 213, 99), (229, 9, 42, 70, 133, 179, 43, 113, 55, 34, 187, 148, 132, 81)\} \cup
\{(229, 4, 94, 216, 100), (229, 10, 44, 71, 135, 180, 45, 114, 57, 35, 189, 149, 134, 82)\} \cup
\{(229, 5, 94, 219, 101), (229, 11, 46, 72, 137, 181, 47, 115, 59, 36, 191, 150, 136, 83)\} \cup
\{(229, 6, 96, 222, 102), (229, 12, 48, 73, 139, 182, 49, 116, 61, 37, 193, 151, 138, 84)\}.
\]
This starter comprises a balanced \((C_5, C_{14})\)-12-foil decomposition of \(K_{229}\).

参考文献