PAPER

Eigenvalue equation of optical transmission fiber considering cross-phase modulation（XPM）induced instability

Wataru Imajuku[1)]

■**Abstract**

The capacity of wavelength division multiplexing（WDM）based optical fiber transmission systems has been limited mainly by the noise power of optical signal and the degree of the distortion of optical signal pulse form. Specifically, nonlinear phase noise which is one of the noise power source has become the most important limiting factor of the WDM based coherent optical transmission systems. And, it is well-known that the source of the nonlinear phase noise power is cross-phase modulation（XPM）interaction occurring amongst the optical signals divided in wavelength space. This paper provides an eigenvalue equation of optical transmission fibers including the the XPM-induced interaction for the first time. This achievement paves the way to establish semi-analytical system design methodology and realize phase noise tolerant optical fiber transmission systems.

波長分割多重光ファイバ伝送システムの容量は、主に光信号の雑音電力と光信号パルス波形の歪の度合いにより制限されており、特に最新の波長分割多重光ファイバ伝送システムでは、光信号の雑音電力源の一種である非線形位相雑音が最も重要な制限要因となっている。そして、この非線形位相雑音の起源は、波長空間において分割された複数の光信号間で発生する相互作用の一つである相互位相変調効果にあることがよく知られている。本論文は、相互位相変調効果を考慮に入れた光伝送ファイバの固有方程式を初めて明らかにする。本論文の成果は、準解析的なシステム設計手法を確立し、位相雑音耐力の高い光ファイバ伝送システムを実現する道筋を与えるものである。

Key Words: Optical fiber, coherent optical fiber transmission, cross-phase modulation, eigenvalue equation

キーワード：光ファイバ、コヒーレント光ファイバ伝送、相互位相変調、固有方程式

1. Introduction

The capacity of wavelength division multiplexing（WDM）based optical fiber transmission systems has been limited mainly by the noise power of optical signal and the degree of the distortion of optical signal pulse form. Huge amount of research and development efforts related to this technological area have been devoted to combat with those two factors for more than several 30 years. The recent invention and commercialization of digital coherent detection technology[1] greatly reduced the limiting factor originating from the latter one. This technology opened new vista to compensate optical signal pulse form distortion caused in optical transmission fiber by applying digital matching filter at optical receiver. As a result, the major limiting factor of the state-of-the art“coherent”optical fiber transmission systems has become the noise power of optical signals. The coherent optical fiber transmission systems employ optical phase modulation format. Therefore, the optical phase noise power is dominant factor to determine the system performance of those systems[2].

The optical phase noise originates from quantum nature of optical signals. It is well known that laser light source can generate almost ideal“coherent”state in the standard quantum limit and make the detector more high sensitive by employing appropriate local oscillator. On the other hand, it is also well-known that the existence of nonlinear refractive index in the optical transmission fiber induces number of nonlinear effects including the“nonlinear”phase noise. The nonlinear phase noise is the converted intensity noise components via the nonlinear refractive index in the optical transmission fiber and the phase noise power can be augmented exponentially in some conditions[3]. In addition,

1）Faculty of Humanity-Oriented Science and Engineering, Kindai University

the nonlinear phase noise power augmentation can be induced by the intensity noise of other optical signal channels in the WDM based optical fiber transmission systems. This is one of inter-channel nonlinear coupling within the optical transmission fiber and known as cross-phase modulation（XPM）[4]. Thus, the nonlinear refractive index of optical transmission fiber induces and augments the nonlinear phase noise power as the propagation of optical signals with the optical transmission fibers.

While number of methodologies have been proposed to mitigate the degradation of the system performance caused by the nonlinear phase noise[5]-[8], there is strong industrial demand developing analytical or semi- analytical model to understand characteristics of the nonlinear phase noise to help designers of the optical transmission systems both in telecom carriers and system vendors as well. Recently, Additive White Gaussian Noise（AWGN）model proposed and has been attracted much attention to estimate the nonlinear phase noise power considering the XPM effect[9]-[11]. Although the proposed methodology provides the way to clearly understand the relationship between the nonlinear phase noise power and optical signal intensity in the WDM based optical fiber transmission systems, the methodology does not cover the case of non-uniform noise spectra. The case might occurs, specifically if the nonlinear phase noise augmentation condition is satisfied and so-called modulation instability is triggered within the optical transmission fibers. It is well known that the modulation instability brings destructive impacts on the performance of the optical fiber transmission systems[12].

This paper proposes alternative methodology which allows the estimation of the nonlinear phase noise power even in the case that the modulation instability is triggered. Especially, this paper provides the eigenvalue equation of the optical transmission fiber including the XPM-induced modulation instability for the first time. The formulation will be conducted by applying small signal analysis[13] which linearlizes the nonlinear differential equation of the optical transmission fibers. This approach provides a semi-analytical system design methodology in the process of the coherent optical fiber transmission system construction planning and design and expected to help realizing phase noise tolerant optical fiber transmission systems.

2. Small signal analysis including the XPM-induced modulation instabil- ity

The propagation property of the optical signal electrical field E_s in the optical transmission fibers is governed by the nonlinear Schrödinger equation（NLSE）as

$$\frac{\partial E_s}{\partial z} + \frac{1}{2}\alpha E_s + \frac{1}{v_g}\frac{\partial E_s}{\partial t} + \frac{i}{2}\beta_2\frac{\partial^2 E_s}{\partial t^2} - \frac{1}{6}\beta_3\frac{\partial^3 E_s}{\partial t^3} = i\gamma|E_s|^2 E_s, \tag{2.1}$$

where z is the transmission distance, t is time, α is the fiber loss coefficient, v_g is the group velocity of the optical signals, β_2 is the first-order group velocity dispersion（GVD）, β_3 is the second-order GVD, γ is the fiber nonlinear coefficient defined by $2\pi n_2/\lambda A_{eff}$, n_2 is the nonlinear refractive index, A_{eff} is the effective core area of the fiber, and λ is the wavelength of the optical carrier. The fiber nonlinear coefficient gives rise to a number of nonlinear effects in the optical transmission fibers such as self-phase modulation（SPM）, XPM, and so on.

The XPM-induced modulation instability being focused in this paper can be analyzed by assuming two optical signal channels and the nonlinear interaction between them. In Eq.（2.1）, the electrical field $E_s(z)$ is assumed as the summation of two optical signal channel components with the center optical frequencies of ω_1 and ω_2 as

$$E_s = \frac{1}{2}[E_1 \exp[i(K_1 z - \omega_1 t)] + E_2 \exp[i(K_2 z - \omega_2 t)]] + c.c., \tag{2.2}$$

where c.c. denotes complex conjugate component of the electrical fields. Those two optical signal channel components are denoted as $E_1(z)$ and $E_2(z)$, respectively. Figure 1 shows the relationship between the two optical signal channels in optical frequency domain. Those two optical signal channels are interacted each other via the fiber nonlinear coefficient, i.e.

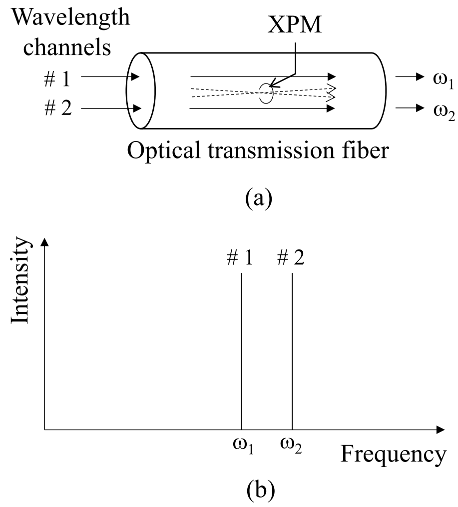

Figure 1

the intensity of the optical signal channel 2 causes the optical phase rotation in the optical signal channel 1 and vice versa.

The strict solution of the NLSE for the two optical signal channels can only be solved by numerical calculation. The Split-Step Fourier Method is most commonly utilized methodology to understand the behavior of the solutions. It requires a number of time consuming studies to understand the tendency of solutions; nevertheless the numerical calculation does not always provide clear understanding of the tendency of solutions. Therefore, this paper tries to introduce the small signal approximation to under the effect of the XPM-induced modulation instability. In the small signal analysis, the optical signal is considered as the summation of the stationary components and the perturbative in-phase and quadrature-phase noise components with the offset optical angular frequency of ω_m as shown in Fig. 2. This model is considered to be applicable for the analysis of the digital coherent transmission systems, since the intensity of the optical signal is constant and the phase modulation format is employed in those systems.

Let small signal analysis be applied for each optical signal channel component and the electrical field amplitude of the signal light with noise at z be

$$E_i(z) = A_{ci}(z) + A_{mi}(z), \tag{2.3}$$

$$A_{mi}(z) = \left(a_i''(z) + ib_i''(z)\right)\exp\left(-i\omega_m t\right) + \left(c_i''(z) - id_i''(z)\right)\exp\left(i\omega_m t\right), \tag{2.4}$$

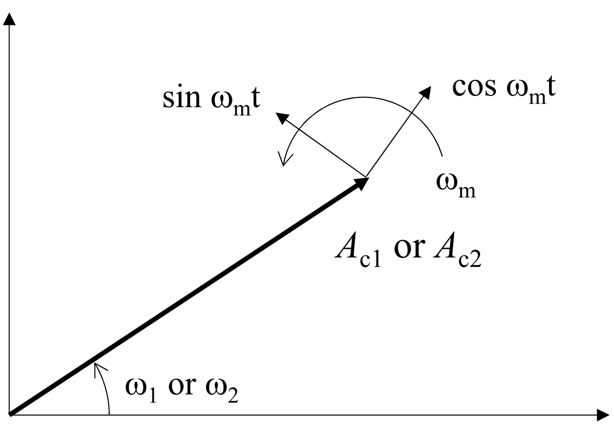

Figure 2

where i is optical signal channel number, A_{ci} is the stationary amplitude of a carrier with the wavelength channel of i, ω is the off-set angular frequency of the noise components (treated as small signal components), and $a_i''(z)$, $b_i''(z)$, $c_i''(z)$, $d_i''(z)$ denote perturbative in-phase and quadrature-phase noise components with the offset optical angular frequency of ω_m from the center frequency of the wavelength channels i.

The substitution of Eqs. (2.2) - (2.4) into Eq. (2.1) leads four frequency components, i.e. exp $[-i\ (\omega_1+\omega_m)t]$, exp$[-i\ (\omega_1-\omega_m)t]$, exp$[-i\ (\omega_2+\omega_m)t]$, and exp$[-i\ (\omega_2-\omega_m)t]$. The separation of real components and imaginary components leads eight simultaneous differential equations as follows:

$$\begin{aligned}
\frac{da_1''(z)}{dz} &= -\frac{1}{2}\alpha a_1''(z) + \frac{\omega_m}{v_{g1}}b_1''(z) - \frac{1}{2}\beta_{21}\omega_m^2 b_1''(z) + \frac{1}{6}\beta_{31}\omega_m^3 b_1''(z) \\
&\quad -2\gamma|A_{c1}|^2 b_1''(z) - \gamma A_{c1}^2 d_1''(z) - 2\gamma A_{c1}(A_{c2}^* b_2''(z) + A_{c2}d_2''(z)), && (2.5)\\
\frac{db_1''(z)}{dz} &= -\frac{1}{2}\alpha b_1''(z) - \frac{\omega_m}{v_{g1}}a_1''(z) + \frac{1}{2}\beta_{21}\omega_m^2 a_1''(z) - \frac{1}{6}\beta_{31}\omega_m^3 a_1''(z) \\
&\quad +2\gamma|A_{c1}|^2 a_1''(z) + \gamma A_{c1}^2 c_1''(z) + 2\gamma A_{c1}(A_{c2}^* a_2''(z) + A_{c2}c_2''(z)), && (2.6)\\
\frac{dc_1''(z)}{dz} &= -\frac{1}{2}\alpha c_1''(z) + \frac{\omega_m}{v_{g1}}d_1''(z) + \frac{1}{2}\beta_{21}\omega_m^2 d_1''(z) + \frac{1}{6}\beta_{31}\omega_m^3 d_1''(z) \\
&\quad +2\gamma|A_{c1}|^2 d_1''(z) + \gamma A_{c1}^2 b_1''(z) + 2\gamma A_{c1}(A_{c2}^* d_2''(z) + A_{c2}b_2''(z)), && (2.7)\\
\frac{dd_1''(z)}{dz} &= -\frac{1}{2}\alpha d_1''(z) - \frac{\omega_m}{v_{g1}}c_1''(z) - \frac{1}{2}\beta_{21}\omega_m^2 c_1''(z) - \frac{1}{6}\beta_{31}\omega_m^3 c_1''(z) \\
&\quad -2\gamma|A_{c1}|^2 c_1''(z) - \gamma A_{c1}^2 a_1''(z) - 2\gamma A_{c1}(A_{c2}^* c_2''(z) + A_{c2}a_2''(z)), && (2.8)\\
\frac{da_2''(z)}{dz} &= -\frac{1}{2}\alpha a_2''(z) + \frac{\omega_m}{v_{g2}}b_2''(z) - \frac{1}{2}\beta_{22}\omega_m^2 b_2''(z) + \frac{1}{6}\beta_{32}\omega_m^3 b_2''(z) \\
&\quad -2\gamma|A_{c2}|^2 b_2''(z) - \gamma A_{c2}^2 d_2''(z) - 2\gamma A_{c2}(A_{c1}^* b_1''(z) + A_{c1}d_1''(z)), && (2.9)\\
\frac{db_2''(z)}{dz} &= -\frac{1}{2}\alpha b_2''(z) - \frac{\omega_m}{v_{g2}}a_2''(z) + \frac{1}{2}\beta_{22}\omega_m^2 a_2''(z) - \frac{1}{6}\beta_{32}\omega_m^3 a_2''(z) \\
&\quad +2\gamma|A_{c2}|^2 a_2''(z) + \gamma A_{c2}^2 c_2''(z) + 2\gamma A_{c2}(A_{c1}^* a_1''(z) + A_{c1}c_1''(z)), && (2.10)\\
\frac{dc_2''(z)}{dz} &= -\frac{1}{2}\alpha c_2''(z) + \frac{\omega_m}{v_{g2}}d_2''(z) + \frac{1}{2}\beta_{22}\omega_m^2 d_2''(z) + \frac{1}{6}\beta_{32}\omega_m^3 d_2''(z) \\
&\quad +2\gamma|A_{c2}|^2 d_2''(z) + \gamma A_{c2}^2 b_2''(z) + 2\gamma A_{c2}(A_{c1}^* d_1''(z) + A_{c1}b_1''(z)), && (2.11)\\
\frac{dd_2''(z)}{dz} &= -\frac{1}{2}\alpha d_2''(z) - \frac{\omega_m}{v_{g2}}c_2''(z) - \frac{1}{2}\beta_{22}\omega_m^2 c_2''(z) - \frac{1}{6}\beta_{32}\omega_m^3 c_2''(z) \\
&\quad -2\gamma|A_{c2}|^2 c_2''(z) - \gamma A_{c2}^2 a_2''(z) - 2\gamma A_{c2}(A_{c1}^* c_1''(z) + A_{c1}a_1''(z)), && (2.12)
\end{aligned}$$

where v_{gi} is the group velocity of the optical signals in wavelength channel i, β_{2i} is the first-order group velocity dispersion (GVD) in wavelength channel i, β_{3i} is the second-order GVD in wavelength channel i. The fiber nonlinear coefficient γ is assumed as the constant value in each wavelength channel.

Now the modulation components in Eq. (2.4) are converted to

$$A_{mi}(z) = (a_i'(z) + ib_i'(z))\cos\omega_m t + (c_i'(z) + id_i'(z))\sin\omega_m t, \tag{2.13}$$

where $a_i'(z)$, $b_i'(z)$, $c_i'(z)$, and $d_i'(z)$ are defined as

$$a_i'(z) \;=\; a_i''(z) + c_i''(z), \tag{2.14}$$

$$b_i'(z) \;=\; b_i''(z) - d_i''(z), \tag{2.15}$$

$$c_i'(z) = -b_i''(z) - d_i''(z), \tag{2.16}$$

$$d_i'(z) = a_i''(z) - c_i''(z). \tag{2.17}$$

The terms of $a_i'(z)$, $c_i'(z)$ are in-phase components and $b_i'(z)$, $d_i'(z)$ are quadrature-phase components. In the digital coherent optical fiber transmission systems, the phase noise power of the optical signals in wavelength channel i detected at an intra-dyne receiver is given as

$$N = 4(< b_i'^2(z) > + < d_i'^2(z) >)E_{LO}^2 B, \tag{2.18}$$

where E_{LO}^2 and B are the local oscillator power and receiver bandwidth at the intra-dyne receiver, respectively.

Therefore, the signal-to-noise power ratio (SNR) at the intra-dyne receiver is given as

$$SNR = \frac{S}{N} = \frac{|A_{ci}|^2 E_{LO}^2}{4(< b_i'^2(z) > + < d_i'^2(z) >)E_{LO}^2 B} = \frac{|A_{ci}|^2}{4(< b_i'^2(z) > + < d_i'^2(z) >)B}. \tag{2.19}$$

Thus, the quadrature-phase noise components $b_i'(z)$, $d_i'(z)$ should be calculated to obtain the phase noise power and the SNR at the intra-dyne receiver.

After the conversion of variables in Eqs. (2.14)-(2.17), the eight simultaneous linear differential equations of Eqs. (2.5)-(2.12) can be greatly simplified as

$$\begin{aligned}
\frac{da_i(z)}{dz} &= -\frac{1}{2}\beta_{2i}\omega_m^2 b_i(z) - \left(\frac{\omega_m}{v_{gi}} + \frac{1}{6}\beta_{3i}\omega_m^3\right) c_i(z) \\
&\quad - \gamma(|\bar{A}_{ci}|^2 - \bar{A}_{ci}^2)b_i(z) - 2\gamma\bar{A}_{ci}(\bar{A}_{cj}^* - \bar{A}_{cj})b_j(z), && (2.20)\\
\frac{db_i(z)}{dz} &= \frac{1}{2}\beta_{2i}\omega_m^2 a_i(z) - \left(\frac{\omega_m}{v_{gi}} + \frac{1}{6}\beta_{3i}\omega_m^3\right) d_i(z) \\
&\quad + \gamma(|\bar{A}_{ci}|^2 + \bar{A}_{ci}^2)a_i(z) + 2\gamma\bar{A}_{ci}(\bar{A}_{cj}^* + \bar{A}_{cj})a_j(z), && (2.21)\\
\frac{dc_i(z)}{dz} &= -\frac{1}{2}\beta_{2i}\omega_m^2 d_i(z) + \left(\frac{\omega_m}{v_{gi}} + \frac{1}{6}\beta_{3i}\omega_m^3\right) a_i(z) \\
&\quad - \gamma(|\bar{A}_{ci}|^2 - \bar{A}_{ci}^2)d_i(z) - 2\gamma\bar{A}_{ci}(\bar{A}_{cj}^* - \bar{A}_{cj})d_j(z), && (2.22)\\
\frac{dd_i(z)}{dz} &= \frac{1}{2}\beta_{2i}\omega_m^2 c_i(z) + \left(\frac{\omega_m}{v_{gi}} + \frac{1}{6}\beta_{3i}\omega_m^3\right) b_i(z) \\
&\quad + \gamma(|\bar{A}_{ci}|^2 + \bar{A}_{ci}^2)c_i(z) + 2\gamma\bar{A}_{ci}(\bar{A}_{cj}^* + \bar{A}_{cj})c_j(z), && (2.23)
\end{aligned}$$

by introducing variables defined as :

$$a_i(z) + b_i(z) + c_i(z) + d_i'(z) = (a'(z) + b'(z) + c'(z) + d'(z))e^{\alpha z/2 - i\gamma P_i z}, \tag{2.24}$$

$$P_i = |\bar{A}_{ci}|^2 = \frac{z'_{eff}}{z'}|A_{ci}|^2 = \frac{1 - e^{-\alpha z'}}{\alpha}\frac{1}{z'}|A_{ci}|^2. \tag{2.25}$$

Here, i and j denote wavelength channel number of the optical signals. The variable i takes the value of 1 or 2, while j takes the value of 2 or 1, respectively.

The eight simultaneous linear differential equation can be expressed in the form of a matrix

$$[M]\begin{bmatrix} a_1(z) \\ b_1(z) \\ c_1(z) \\ d_1(z) \\ a_2(z) \\ b_2(z) \\ c_2(z) \\ d_2(z) \end{bmatrix} = \begin{bmatrix} D & \eta_1 & \delta_1 & 0 & 0 & \sigma_1 & 0 & 0 \\ -\varepsilon_1 & D & 0 & \delta_1 & -\rho_1 & 0 & 0 & 0 \\ -\delta_1 & 0 & D & \eta_1 & 0 & 0 & 0 & \sigma_1 \\ 0 & -\delta_1 & -\varepsilon_1 & D & 0 & 0 & -\rho_1 & 0 \\ 0 & \sigma_2 & 0 & 0 & D & \eta_2 & \delta_2 & 0 \\ -\rho_2 & 0 & 0 & 0 & -\varepsilon_2 & D & 0 & \delta_2 \\ 0 & 0 & 0 & \sigma_2 & -\delta_2 & 0 & D & \eta_2 \\ 0 & 0 & -\rho_2 & 0 & 0 & -\delta_2 & -\varepsilon_2 & D \end{bmatrix} \begin{bmatrix} a_1(z) \\ b_1(z) \\ c_1(z) \\ d_1(z) \\ a_2(z) \\ b_2(z) \\ c_2(z) \\ d_2(z) \end{bmatrix} = 0, \tag{2.26}$$

where the differential operator D is introduced and the parameters are defined as:

$$D = \frac{d}{dt}, \tag{2.27}$$

$$\varepsilon_i = \frac{\beta_{2i}\omega_m^2}{2} + \gamma \bar{A}_{ci}(\bar{A}_{ci}^* + \bar{A}_{ci}), \tag{2.28}$$

$$\eta_i = \frac{\beta_{2i}\omega_m^2}{2} + \gamma \bar{A}_{ci}(\bar{A}_{ci}^* - \bar{A}_{ci}), \tag{2.29}$$

$$\delta_i = \frac{\omega_m}{v_{gi}} + \frac{\beta_{3i}\omega_m^3}{6}, \tag{2.30}$$

$$\rho_i = 2\gamma \bar{A}_{ci}(\bar{A}_{cj}^* + \bar{A}_{cj}), \tag{2.31}$$

$$\sigma_i = 2\gamma \bar{A}_{ci}(\bar{A}_{cj}^* - \bar{A}_{cj}). \tag{2.32}$$

16

The terms including ρ_i and/or σ_i indicates the coupling of wavelength channel i and j. The XPM is incorporated in those terms.

The further simplification can be achieved if the argument of A_{ci} is considered as 0 [rad] and the average power of the optical signals in each wavelength channel is the same, i.e. $|\bar{A}_{c1}| = |\bar{A}_{c2}|$. Equations (2.28), (2.29), (2.31) and (2.32) become

$$\varepsilon_i = \frac{\beta_{2i}\omega_m^2}{2} + 2\gamma P_i, \tag{2.33}$$

$$\eta_i = \frac{\beta_{2i}\omega_m^2}{2}, \tag{2.34}$$

$$\rho_i = \rho_j = 4\gamma P_i = 4\gamma P_j = \rho, \tag{2.35}$$

$$\sigma_i = 0. \tag{2.36}$$

Thus, Eq. (2.26) can be further simplified as

$$[M]\begin{bmatrix} a_1(z) \\ b_1(z) \\ c_1(z) \\ d_1(z) \\ a_2(z) \\ b_2(z) \\ c_2(z) \\ d_2(z) \end{bmatrix} = \begin{bmatrix} D & \eta_1 & \delta_1 & 0 & 0 & 0 & 0 & 0 \\ -\varepsilon_1 & D & 0 & \delta_1 & -\rho & 0 & 0 & 0 \\ -\delta_1 & 0 & D & \eta_1 & 0 & 0 & 0 & 0 \\ 0 & -\delta_1 & -\varepsilon_1 & D & 0 & 0 & -\rho & 0 \\ 0 & 0 & 0 & 0 & D & \eta_2 & \delta_2 & 0 \\ -\rho & 0 & 0 & 0 & -\varepsilon_2 & D & 0 & \delta_2 \\ 0 & 0 & 0 & 0 & -\delta_2 & 0 & D & \eta_2 \\ 0 & 0 & -\rho & 0 & 0 & -\delta_2 & -\varepsilon_2 & D \end{bmatrix} \begin{bmatrix} a_1(z) \\ b_1(z) \\ c_1(z) \\ d_1(z) \\ a_2(z) \\ b_2(z) \\ c_2(z) \\ d_2(z) \end{bmatrix} = 0. \tag{2.37}$$

The solution of $a_1(z)$, $b_1(z)$, $c_1(z)$, $d_1(z)$, $a_2(z)$, $b_2(z)$, $c_2(z)$, $d_2(z)$ includes linearly coupled eight terms respectively and each term is given in the form of

$$f(z) = c_{xi}\exp(\lambda z) = c_{xi}\exp(iKz), \tag{2.38}$$

where λ and K are the eigenvalue of the eight simultaneous linear differential equation, c_{xi} is the constant value determined by boundary conditions, and the value of c_{xi} differs term-by-term basis in general.

Therefore, the differential operator D in Eq. (2.37) can be replaced by iK and the determinant of the matrix $[M]$ is forced to satisfy

$$|M| = \begin{vmatrix} iK & \eta_1 & \delta_1 & 0 & 0 & 0 & 0 & 0 \\ -\varepsilon_1 & iK & 0 & \delta_1 & -\rho_1 & 0 & 0 & 0 \\ -\delta_1 & 0 & iK & \eta_1 & 0 & 0 & 0 & 0 \\ 0 & -\delta_1 & -\varepsilon_1 & iK & 0 & 0 & -\rho_1 & 0 \\ 0 & 0 & 0 & 0 & iK & \eta_2 & \delta_2 & 0 \\ -\rho_2 & 0 & 0 & 0 & -\varepsilon_2 & iK & 0 & \delta_2 \\ 0 & 0 & 0 & 0 & -\delta_2 & 0 & iK & \eta_2 \\ 0 & 0 & -\rho_2 & 0 & 0 & -\delta_2 & -\varepsilon_2 & iK \end{vmatrix} = 0. \tag{2.39}$$

The further calculation of Eq. (2.39) leads

$$\left[(K^2 - \varepsilon_1\eta_1 - \delta_1^2)^2 - 4\varepsilon_1\eta_1\delta_1^2\right]\left[(K^2 - \varepsilon_2\eta_2 - \delta_2^2)^2 - 4\varepsilon_2\eta_2\delta_2^2\right] = C_{XPM}^2, \tag{2.40}$$

$$C_{XPM}^2 = \eta_1^2\eta_2^2\rho^4 + 4\eta_1\eta_2\delta_1\delta_2\rho^2\left[(\delta_1+\delta_2)^2 - (\delta_1+\delta_2-\sqrt{\varepsilon_1\eta_1})^2 - (\delta_1+\delta_2-\sqrt{\varepsilon_2\eta_2})^2\right]. \tag{2.41}$$

Those are the analytical eigenvalue equation of the optical transmission fiber taking into account the XPM-induced modulation instability. The first element of left-hand side term in Eq. (2.40) is the eigenvalue equation for wavelength channel i =1. The second element of left-hand side term in Eq. (2.40) is the eigenvalue equation for wavelength channel i =2. C_{XPM} is the terms indicating the coupling of wavelength channel 1 and 2. K has eight solutions and those are considered as the eigenvalues of the optical transmission fiber taking into account the XPM-induced modulation instability. The analytical solution of K provides the analytical noise component functions, $a_1(z)$, $b_1(z)$, $c_1(z)$, $d_1(z)$, $a_2(z)$, $b_2(z)$, $c_2(z)$, $d_2(z)$, which allows obtaining analytical transfer matrix of the optical transmission fiber including the XPM-induced modulation instability[14]. The implication of the obtained analytical eigenvalue equation will be evaluated and discussed in the next section.

3. Implication of the eigenvalue equation including XPM-induced mod-ulation instability

The analytical eigenvalue equation of Eq. (2.40) obtained in former section is eighth order linear algebraic equation of K. Therefore, the eigenvalue equation not always provide analytical solution of K according to algebraic theory. However, the analytical solution can be obtained in some conditions and it is intuitive to evaluate the solution in the conditions.

3.1 The case without XPM-induced modulation instability

As a first step, this paper addresses the case without the XPM-induced modulation instability. In this case, the value of ρ is 0 and the analytical eigenvalue equation of (2.40) becomes

$$\left[(K^2-\varepsilon_1\eta_1-\delta_1^2)^2-4\varepsilon_1\eta_1\delta_1^2\right]\left[(K^2-\varepsilon_2\eta_2-\delta_2^2)^2-4\varepsilon_2\eta_2\delta_2^2\right]=0. \tag{3.1}$$

Thus, it is obvious that the eigenvalue equation can be decoupled into two components. The solution of K for the optical signal in wavelength channel i is given as

$$K=\pm(\sqrt{\varepsilon_1\eta_i}-\delta_1),\pm(\sqrt{\varepsilon_1\eta_1}+\delta_1),\pm(\sqrt{\varepsilon_2\eta_i}-\delta_2),\pm(\sqrt{\varepsilon_2\eta_2}+\delta_2). \tag{3.2}$$

Therefore, the noise component functions, $a_1(z)$, $b_1(z)$, $c_1(z)$, $d_1(z)$, $a_2(z)$, $b_2(z)$, $c_2(z)$, $d_2(z)$ are also decou-pled per channel basis as well. Thus, the noise component functions are given as :

$$\begin{bmatrix} a_i(z)\\ b_i(z)\\ c_i(z)\\ d_i(z)\end{bmatrix}=\begin{bmatrix}
\cos(\sqrt{\varepsilon_i\eta_i}z)\cos(\delta_i z) & -\sqrt{\eta_i/\varepsilon_i}\sin(\sqrt{\varepsilon_i\eta_i}z)\cos(\delta_i z) & -\cos(\sqrt{\varepsilon_i\eta_i}z)\sin(\delta_i z) & \sqrt{\eta_i/\varepsilon_i}\sin(\sqrt{\varepsilon_i\eta_i}z)\sin(\delta_i z)\\
\sqrt{\varepsilon_i/\eta_i}\sin(\sqrt{\varepsilon_i\eta_i}z)\cos(\delta_i z) & \cos(\sqrt{\varepsilon_i\eta_i}z)\cos(\delta_i z) & -\sqrt{\varepsilon_i/\eta_i}\sin(\sqrt{\varepsilon_i\eta_i}z)\sin(\delta_i z) & -\cos(\sqrt{\varepsilon_i\eta_i}z)\sin(\delta_i z)\\
\cos(\sqrt{\varepsilon_i\eta_i}z)\sin(\delta_i z) & -\sqrt{\eta_i/\varepsilon_i}\sin(\sqrt{\varepsilon_i\eta_i}z)\sin(\delta_i z) & \cos(\sqrt{\varepsilon_i\eta_i}z)\cos(\delta_i z) & -\sqrt{\eta_i/\varepsilon_i}\sin(\sqrt{\varepsilon_i\eta_i}z)\cos(\delta_i z)\\
\sqrt{\varepsilon_i/\eta_i}\sin(\sqrt{\varepsilon_i\eta_i}z)\sin(\delta_i z) & \cos(\sqrt{\varepsilon_i\eta_i}z)\sin(\delta_i z) & \sqrt{\varepsilon_i/\eta_i}\sin(\sqrt{\varepsilon_i\eta_i}z)\cos(\delta_i z) & \cos(\sqrt{\varepsilon_i\eta_i}z)\cos(\delta_i z)
\end{bmatrix}\begin{bmatrix} a_i(0)\\ b_i(0)\\ c_i(0)\\ d_i(0)\end{bmatrix}, \tag{3.3}$$

where the boundary conditions of the noise components in wavelength channel i at z = 0 are assumed as $a_i(0)$, $b_i(0)$, $c_i(0)$, $d_i(0)$ and i takes the value of 1 or 2. The solution of the eigenvalue equation provides analytical transfer function of the noise components for the optical transmission fiber.

3.2 The case with zero GVD

This case makes β_{21} = β_{22} = 0 and greatly simplifies the the analytical eigenvalue equation of Eq. (2.40). The further calculation leads

$$\left[(K^2-\delta_1^2)^2\right]\left[(K^2-\delta_2^2)^2\right]=0. \tag{3.4}$$

The solution of K for the optical signals in wavelength channel i is given as doubly degenerated value with

$$K=\pm\delta_1,\pm\delta_2. \tag{3.5}$$

3.3 The case without the group-velocity mismatch and second-order GVD

The analytical eigenvalue equation of Eq. (2.40) can be greatly simplified in the case that the group- velocity mismatch between wavelength channels can be ignored and the second-order GVD, β_{3i} is 0.

In this condition, time t can be replaced by relative time T which has relationship of

$$T=t-\frac{z}{v_{g1}}=t-\frac{z}{v_{g2}}. \tag{3.6}$$

Let the electrical field amplitude of the signal light with noise at z be

$$A_{mi}(z)=(a_i'(z)+ib_i'(z))\cos\omega_m T+(c_i'(z)+id_i'(z))\sin\omega_m T. \tag{3.7}$$

This case derives the condition of δ_1 = δ_2 =0 in the eigenvalue equation of (2.40). Thus, the eigenvalue equation of Eq. (2.40) becomes

$$\left[(K^2-\varepsilon_1\eta_1)^2\right]\left[(K^2-\varepsilon_2\eta_2)^2\right]=\eta_1^2\eta_2^2\rho^4. \tag{3.8}$$

The solution of the eigenvalue equation can be analytically derived as

$$K=\pm\left[\frac{\varepsilon_1\eta_1+\varepsilon_2\eta_2}{2}\pm\left[\left(\frac{\varepsilon_1\eta_1-\varepsilon_2\eta_2}{2}\right)^2\pm\eta_1\eta_2\rho^2\right]^{1/2}\right]^{1/2}. \tag{3.9}$$

This solution becomes unstable if the variable K becomes imaginary part for some values of ω_m. The noise component functions of $a_1(z)$, $b_1(z)$, $c_1(z)$, $d_1(z)$, $a_2(z)$, $b_2(z)$, $c_2(z)$, $d_2(z)$ experience an exponential growth along the fiber.

In the presence of the XPM coupling, Eq. (3.9) can be unstable, if the condition

$$\frac{\varepsilon_1\eta_1+\varepsilon_2\eta_2}{2}\pm\left[\left(\frac{\varepsilon_1\eta_1-\varepsilon_2\eta_2}{2}\right)^2\pm\eta_1\eta_2\rho^2\right]^{1/2}<0, \tag{3.10}$$

is satisfied. In the case of

$$K=\left[\frac{\varepsilon_1\eta_1+\varepsilon_2\eta_2}{2}+\left[\left(\frac{\varepsilon_1\eta_1-\varepsilon_2\eta_2}{2}\right)^2+\eta_1\eta_2\rho^2\right]^{1/2}\right]^{1/2}, \tag{3.11}$$

Eq. (3.10) is expressed as

$$[\omega_m^2/\omega_{c1}^2+1][\omega_m^2/\omega_{c2}^2+1]<4, \tag{3.12}$$

$$\omega_{ci}=\sqrt{4\gamma P/|\beta_{2i}|}. \tag{3.13}$$

The most important conclusion from Eq. (3.12) is that the XPM-induced modulation instability can occur irrespective of the signs of β_{2i}. Now, let the values of $|\beta_{2i}|$, γ, and P be 10 [ps^2/km], 2.0 [1/km/W] , and 2.0 [mW]. The value of f_{ci} becomes

$$f_{ci}=\omega_{ci}/2\pi=\sqrt{4\gamma P/|\beta_{2i}|}=\sqrt{4\times 2\times\ 0.002/(10\times 10^{-24})}\approx 6\times 10^{-3}\ \ [\mathrm{THz}]. \tag{3.14}$$

Thus, the frequency condition satisfying Eq. (3.12) is in the order of <10 [GHz], if the optical signals are transmitted in the dispersion shifted single mode optical fibers (DSFs). The phase noise bandwidth of the XPM-induced modulation instability cannot be negligible in the case of DSFs. Although the condition is mitigated by half if the WDM-based coherent optical fiber transmission system employs standard single ode fibers (SMFs), the XMP-induced modulation instability can be destructive factor to limit the system performance. Thus, the XPM-induced modulation instability is one of non-negligible factor in the fibers widely utilized globally. The XPM-induced modulation instability need to be addressed in the process of the system design.

4. Conclusion

This paper addressed the nonlinear phase noise including XPM-based modulation instability in the WDM-based coherent optical fiber transmission systems theoretically. Different from conventional approach based on numerical calculation of the NLSE, the small signal analysis successfully provided the set of eight simultaneous linear differential equations. The set of equations forms the eigenvalue equations of the NLSE. The XPM-induced modulation instability was clearly explained by using the obtained eigenvalue equations. The solution of the eigenvalue equation provided intuitive results that the XPM-induced modulation insta bility is the phenomena need to be addressed in the process of the systems design. The instable condition might occur in the frequency range of <10 [GHz] irrespective to the sign of the GVD β_2 of the WDM-based coherent optical fiber transmission systems with the DSFs. The frequency range can be reduced by half in that with the SMFs.

Simultaneously, the results indicate that the importance of developing the methodology to reduce the non- linear phase noise as the author proposed previously in Ref. 8. The author believe the proposed methodology employing Periodically Poled Lithium Niobate（PPLN）waveguides[6],[15] applicable even in the condition that the XPM-based modulation instability is dominant. It is preferable to conduct theoretical study to estimate the impact of the proposed methodology in the case that the XPM-induced modulation instability is dominant. To reach this goal, the noise component functions for the eigenvalues obtained, which was missing portion in this paper, should be addressed.

5. Acknowledgments

This work was supported by JSPS KAKENHI Grant Number JP17K06451.

References

[1] D. -S. Ly-Gagnon, K. Katoh, and K. Kikuchi, "Unrepeatered optical transmission of 20 Gbit/s quadra-ture phase-shift keying signals over 210 km using homodyne phase-diversity receiver and digital signal processing," IET ELECTRONICS LETTERS, vol. 41, No. 4, pp. 206-207. Feb.（2005）.

[2] T. Okoshi, "Ultimate performance of heterodyne/coherent optical fiber communications," IEEE J. Lightwave Technol. vol. 4, No. 10, pp. 1556-1562, Oct.（1986）.

[3] D. Marcuse, "Single-channel operation in very long nonlinear fibers with optical amplifiers at zero dispersion," IEEE J. Lightwave Technol., vol. LT-9, pp. 356-361, Mar.（1991）.

[4] G. P. Agrawal, "Nonlinear fiber optics second edition," Academic Press, New York,（1995）.

[5] A. Takada and W. Imajuku, "Amplitude noise suppression using a high gain phase sensitive amplifier as a limiting amplifier, "IET ELECTRONICS LETTERS, vol. 32, No. 7, pp. 677-678. March（1996）.

[6] A. Bogoni, X. Wu, S. R. Nuccio, and A. E. Willner, " 640 Gb/s All-Optical Regenerator Based on a Periodically Poled Lithium Niobate Waveguide," IEEE J. Lightwave Technol., vol. 30, NO. 12, pp. 1829-1834, June（2012）.

[7] O. Geller, R. Dar, M. Feder, and M. Shtaif, " A Shaping Algorithm for Mitigating Inter-Channel Non- linear Phase-Noise in Nonlinear Fiber Systems," IEEE J. Lightwave Technol., vol. 34, No. 16, Aug.（2016）.

[8] W. Imajuku, K. Nakaishi, M. Kakoi, K. Mori, and M. Fukutoku, " Nonlinear Phase Noise Reduction by Gain Saturated Parametric Amplifiers in Coherent Optical Transmission," Proc. of Nonlinear Optics（NLO）2017, to be appeared, July（2017）.

[9] A. Carena, V. Curri, G. Bosco, P. Poggiolini, and F. Forghieri, " Modeling of the Impact of Nonlinear Propagation Effects in Uncompensated Optical Coherent Transmission Links, " IEEE J. Lightwave Technol., vol. 30, No. 10, pp. 1524-1539, Oct.（2012）.

[10] R. Dar, M. Feder, A. Mecozzi, and M. Shtaif, "Properties of nonlinear noise in long, dispersion- uncompensated fiber links, " Opt. Exp., vol. 21, pp. 25685-25699, Nov.（2013）.

[11] R. Dar, M. Feder, Inter-Channel Nonlinear Interference Noise in WDM Systems: Modeling and Mit-igation, IEEE J. Lightwave Technol., vol. 33, No. 5, pp.1044-1051, March（2015）.

[12] S. Saito and A. Naka, "Noise augmentation and signal-to-noise ratio degradation in in-line amplifier systems due to the nonlinear refractive index of optical fibers," Electronics and Communications in Japan, Part 1, vol. 78, pp. 11-20, Nov.（1995）.

[13] W. Imajuku and A. Takada, "Gain characteristics of coherent optical amplifier using a Mach-Zehnder interferometer with Kerr media," IEEE J. Quantum Electron., vol. QE-35, pp. 1657-1665, Nov.（1999）.

[14] W. Imajuku and A. Takada, "Theoretical Analysis of System Limitation for AM-DD/NRZ Optical Transmission Systems Using In-Line Phase-Sensitive Amplifiers," IEEE J. Lightwave Technol., vol. 16, No. 7, pp. 1158-1170, July（1998）.

[15] T. Umeki, O. Tadanaga, and M. Asobe, "Highly Efficient Wavelength Converter Using Direct-Bonded PPZnLN Ridge Waveguide," IEEE J. Quantum. Electron., vol. 46, No. 8, pp. 1206-1213, Aug.（2010）.