Public Capital, Capacity Utilization, and Economic Growth

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Abstract　This paper incorporates capital utilization in an endogenous growth model with public capital, and examines the effects of fiscal policy on both economic performance and welfare.　Dynamic analysis reveals that maximizing equilibrium capacity coincides with maximizing the economic growth rate in the long-run, though not in the short-run.　It also demonstrates that the growth-maximizing tax rate is either increasing or decreasing with respect to the marginal cost of private capital utilization and capacity utilization of public capital depending on the elasticity of substitution. Welfare analysis shows that the growth-maximizing government not only over-invests but also under-invests in public capital stock.　Similarly, it shows not only an excess use of public capital but also insufficient use of public capital.

1　Introduction

The effects of public investment on economic performance and optimal fiscal policy have long been the focus of numerous theoretical and empirical studies. In the early dynamic general equilibrium models developed by Arrow and Kurz (1970) and subsequent studies, public capital stock as a sequel to accumulated public investment is incorporated into one of the key factors of production. Numerous empirical studies find that public capital produces a significant growth effect (e.g. Aschauer 1989; Munnell 1992; Gramlich 1994; Kneller et al. 1999).[1]

In later eras, Futagami et al. (1993), Glomm and Ravikumar (1994), and Fisher and Turnovsky (1998) among others developed endogenous growth models with public capital on the basis of the model presented by Barro (1990) and additional empirical evidences.[2] These models have been widely used for further analyses of the effect of public investment on macroeconomic performance by applying new ideas to real problems.[3] Particularly, Rioja (2003) and Kalaitzidakis and Kalyvitis (2004) incorporated the concept of maintenance in public capital to an endogenous growth model including public capital.[4]

In fact, in many countries, awareness of the importance of public capital maintenance has grown over the last several decades. Rioja (2003) and Kalaitzidakis and Kalyvitis (2004) have shed light on the trade-off between new and replacement investment in public capital by analyzing the case of maintenance expenditure, which affects public capital's depreciation rate. In many cases, the main factor for public capital maintenance is aged deterioration of infrastructure, and it is

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(1) Pereira and Andraz (2012) surveyed recent empirical studies on this topic.

(2) See Irmen and Kuehnel (2009) for a general review of the applications of Barro's (1990) model.

(3) A large number of studies exist: Greiner and Hanusch (1998) studied the growth and welfare effects of public investment, investment subsidies and redistributive transfers; Yakita (2004) incorporated monopolistic competition in an endogenous growth model with public capital; Greiner (2007) investigated the issue on sustainable government debt.

(4) More recently, Dioikitopoulos and Kalyvitis (2008) incorporated a congestion effect and study how it affect optimal and growth-maximizing fiscal policies.

natural to consider such deterioration of public capital in proportion to its frequency of use.

The concept of user cost as noted by Keynes (1936) involves capacity utilization of equipment. Concerning the accumulation of private capital, many studies incorporate user costs in the sense that a higher utilization rate causes faster capital stock depreciation the models (e.g. Calvo 1975; Greenwood et al. 1988; Chatterjee 2005). This idea is applicable to the accumulation of public capital and is important to the investigation of the relation between economic growth and private and public capital services.

In their recent, Chatterjee and Mahbub Morshed (2011) study the difference between private and government provision of infrastructure using an endogenous growth model with endogenous capital utilization. In their model, capital utilization causes different effects on market prices for capital goods under each of the two infrastructure provision regimes as well as causing different fiscal policy effects on economic performance through different transmission mechanisms. They also show that the choice between private and government provision is key to designing an optimal fiscal policy.

By contrast, this paper provides comprehensive analyses of the interaction between capital utilization, production structure, fiscal policy, and economic performance and studies how difference in government policy targets such as growth and welfare maximization, affect economic performance. The significant feature that differentiates our analysis from the existing literature is our focus on the general class of production technology to emphasize the substitutability or complementarity between private and public capital services.

The need to make a capital utilization decision then drives a flexible production schedule through a flexible change in the ratio of private to public capital service in response to cost and policy change. Then, the question becomes whether the substitutability or complementarity between private and public capital service imparts different impacts via a change in the private to public capital service ratio. This issue is particularly insightful for investigating into the productivity effect of public capital based on empirical evidence such as Seitz (1994), Nadiri and Mamuneas

(1994), Cohen and Morrison Paul (2004) and Vijverberg and Vijverberg (2007).[5]

In keeping with the above intention, this paper firstly analyzes the long-run effect on economic performance of fiscal policy financed by income tax. We find that the tax rate, which maximizes the equilibrium utilization rate of private capital, is equivalent to the growth-maximizing tax rate in the long-run, while the cost of private capital utilization affects the growth-maximizing tax rate according to the elasticity of input substitution. The comparative dynamics analysis derives that the transitional dynamics after a tax rate rise is characterized by instantaneous negative effects on both the utilization rate of private capital and ratio of private to public capital service. Meanwhile, the instantaneous effect on consumption results from the income and intertemporal substitution effect. Thus, the short-run effects on growth rates depends upon the relative magnitude of these effects.

We also conduct a welfare analysis of fiscal policy financed by income tax and find that whether the welfare-maximizing tax rate is higher or lower than the growth-maximizing tax rate depends upon the relative magnitude of the instantaneous effect on consumption and transitional effect of consumption growth. When faced with a capital utilization decision, a growth-maximizing government has the incentive to not only over-invest, but to under-invest in public capital stock. The result of these conflicting pressures adds new perspectives to the relation between welfare- and growth-maximizing fiscal policy with existing studies on the effects of fiscal policy.

Subsequently, this paper analyzes the effect of public capacity utilization on economic performance in both the short- and long-run. Our analysis shows that the utilization rate of public capital, which maximizes the equilibrium utilization rate of private capital, equals the growth-maximizing utilization rate in the long-run. Furthermore, we demonstrate that an increase in the private capital utilization costs reduces the growthmaximizing utilization rate of public capital and that a

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(5) Recently, some empirical studies showed that the elasticity of substitution between the factors of production including factors augmenting technological progress, does not equal unity (e.g., Klump et al. 2007; León-Ledesma et al. 2010). The Cobb-Douglas production function, which is widely used in the economic growth theory, is not supported.

rise in the public capital utilization rate will either raise or reduce the growth-maximizing tax rate according to the elasticity of input substitution.

Finally, we use comparative dynamics analysis to examine the welfare effect of public capacity utilization and show that the welfare-maximizing utilization rate of public capital is either higher or lower than the growth-maximizing utilization rate depending on the relative magnitude of the instantaneous effect on consumption and the transitional effect of consumption growth. Under a growth-maximizing government, the producer uses public capital not only excessively but also insufficiently. Our analysis provides a comprehensive analysis of fiscal policy under a capital utilization decision, which complements existing studies on the effects of fiscal policy.

The remainder of this paper is organized as follows: Section 2 presents a description of our model, solves the model, and characterizes the transitional dynamics. Section 3 investigates the dynamic interaction between taxation, economic growth, and welfare. Section 4 presents a dynamic analysis of the relationship between public capacity utilization, economic growth, and welfare. Finally, Section 5 concludes this paper.

2   The model

2.1   Basic setup

Consider a closed economy with a single final good and two capital input service. The economy consists of identical rational households with infinite planning horizons. The population is normalized to unity. The output of the final good is determined by the production function $Y_t = F(Z_t, P_t)$ where Z_t is private capital service and P_t is public capital service. Let u_t be the utilization rate of private capital, K_t be the private capital stock, v_t be the utilization rate of public capital and G_t be the public capital stock. Both private capital service and public capital service are defined as $Z_t := u_t K_t$ and $P_t := v_t G_t$, respectively.

The capital utilization decision incurs a user cost because of which a higher utilization rate brings about faster capital stock depreciation. Following Calvo

(1975) and Greenwood et al. (1988), we introduce this effect into our model as the depreciation functions $\delta(u_t)$ and $\eta(v_t)$. That is, $\delta(u_t)$ is the depreciation rate of private capital where $\delta'(\cdot) > 0$ and $\delta''(\cdot) > 0$, and $\eta(v_t)$ is the depreciation rate of public capital where $\eta'(\cdot) > 0$ and $\eta''(\cdot) > 0$.[6] Taking this into account, the evolution of private and public capital stocks are $\dot{K}_t = I_t - \delta(u_t)K_t$ and $\dot{G}_t = H_t - \eta(v_t)G_t$ where I_t is investment in private capital stock and H_t is investment in public capital stock.[7]

Assume that the production function F satisfies a constant returns to scale. Then, $f(x_t) := F(x_t, 1)$ where $x_t := Z_t/P_t = (u_t K_t)/(v_t G_t)$ is the ratio of private to public capital service. We also assume that $f'(x_t) > 0$ and $f''(x_t) < 0$. For use in subsequent analyses, we define the output elasticity of public capital service as $\alpha(x_t) := 1 - x_t f'(x_t)/f(x_t)$ and elasticity of the marginal product of private capital service with respect to x_t as $\phi(x_t) := -x_t f''(x_t)/f'(x_t)$.[8]

Then, the elasticity of substitution can be given as

$$\epsilon(x_t) = \frac{1 - \alpha(x_t)}{1 - \alpha(x_t) - \varphi(x_t)} = \frac{\alpha(x_t)}{\phi(x_t)}, \tag{1}$$

where $\varphi(x_t) := x_t \alpha'(x_t)/\alpha(x_t)$ denotes the second-order output elasticity of public capital service.[9] If this second-order elasticity is positive or if the output elasticity of public capital service is increasing at x_t, then the elasticity of input substitution is larger than unity, that is, private capital service is a substitute for public capital service. If the second-order elasticity is negative or if the output elasticity of public capital service is decreasing at x_t, then the elasticity of input substitution is smaller than unity, that is, private capital service is complementary to public capi-

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(6) Rioja (2003) and Kalaitzidakis and Kalyvitis (2004) assumed that the public capital depreciation rate depends on the ratio of maintenance expenditure to aggregate output.

(7) Chatterjee and Mahbub Morshed (2011) incorporated an adjustment cost of investment to analyze the private provision of infrastructure because an adjustment cost yields the explicit evolution of capital price.

(8) By the properties of the production function, $0 \leq \alpha(x_t) \leq 1$ holds. Furthermore, the output elasticity of private capital service is defined as $1 - \alpha(x_t)$.

(9) Note that we have
$$\phi(x_t) = \left[\alpha(x_t) - \frac{x_t \alpha'(x_t)}{1 - \alpha(x_t)}\right].$$

tal service. As a result, ϕ is related to φ and ϵ.

The representative household allocates its net income for its consumption expenditure and savings (investment in private capital). Accordingly, the households have the following budget constraint:

$$\dot{K}_t = (1 - \tau)Y_t - C_t - \delta(u_t)K_t, \tag{2}$$

where C_t is private consumption and τ is the income tax rate.

Households choose both their amount of private consumption and their utilization rate of private capital to maximize their lifetime utility function subject to their budget constraints. We can consider the case that the household chooses the utilization rate of public capital for given stock of public capital. However, in such case, the households set the utilization rate of public capital to its maximum level because they do not have to pay the user cost. Therefore, the household's optimization problem is formalized as

$$\max_{C_t, u_t} \int_0^\infty \frac{C_t^{1-\theta} - 1}{1 - \theta} e^{-\rho t} dt,$$

subject to (2) taking K_0 and the evolution of P_t as given. Solving the optimization problem, we obtain

$$\frac{\dot{C}_t}{C_t} = \frac{(1 - \tau)u_t f'(x_t) - \delta(u_t) - \rho}{\theta}, \tag{3}$$

$$(1 - \tau)f'(x_t) = \delta'(u_t), \tag{4}$$

as well as the transversality condition. Equation (3) is a well-known condition called the Euler equation. Equation (4) is the first-order condition for the optimal utilization rate of private capital: the marginal cost of private capital utilization should be equal to the net marginal product of private capital service that corresponds to the marginal benefit of private capital utilization.

We can explain the government's provision of public capital service. Government

taxes household income and allocates tax revenues to public capital investment. Following Futagami et al. (1993), we assume that the incremental quantity of public capital stock equals the net investment in public capital.　Since we focus on the capital capacity of private capital as well as public capital, the depreciation rate of public capital depends upon its utilization rate.　Accordingly, the government's budget constraint combined with the evolution of public capital stock becomes

$$\dot{G}_t = \tau Y_t - \eta(v_t)G_t. \tag{5}$$

As already mentioned, the households desire full utilization of public capital.　It is important to consider the supremum of utilization rate of public capital.　The user cost of public capital is financed by the government.　Therefore, we assume that the government sets the public capital utilization rate to a positive constant level ν: $v_t = \nu$.

By equation (4) and $v_t = \nu$, the utilization rate of private capital and ratio of private capital service to public capital service are function with respect to the ratio of private capital stock to public capital stock and exogenous variables such as τ and ν.　Let $k_t := K_t/G_t$ as the ratio of private to public capital stock and $n_t := u_t/v_t$ as the ratio of private capital's utilization rate to public capital's utilization rate. Then, we have $x_t \equiv n_t k_t$.

Using (4) and $x_t \equiv n_t k_t$, we obtain $u_t = u(k_t, \nu, \tau)$ such as

$$\frac{\partial u_t}{\partial k_t} = \frac{u_t(1-\tau)f''(x_t)}{v_t\delta''(u_t) - k_t(1-\tau)f''(x_t)} < 0,$$

$$\frac{\partial u_t}{\partial \tau} = -\frac{v_t f'(x_t)}{v_t\delta''(u_t) - k_t(1-\tau)f''(x_t)} < 0,$$

$$\frac{\partial u_t}{\partial \nu} = -\frac{(1-\tau)x_t f''(x_t)}{v_t\delta''(u_t) - k_t(1-\tau)f''(x_t)} > 0.$$

Recall equation (4).　An increase in k_t decreases the net marginal product of private capital service and also decreases the marginal cost of private capital utilization. Therefore, an increase in k_t reduces the equilibrium utilization rate of private capital.

This mechanism is similar to the situation in which τ rises. On the other hand, a rise in ν brings about the opposite outcome because it increases the net marginal product of private capital service.

Furthermore, using a method similar to that used to derive the properties of $u_t = u(k_t, \nu, \tau)$, we also get $x_t = x(k_t, \nu, \tau)$ such as

$$\frac{\partial x_t}{\partial k_t} = \frac{u_t \delta''(u_t)}{v_t \delta''(u_t) - k_t(1-\tau)f''(x_t)} > 0,$$

$$\frac{\partial x_t}{\partial \tau} = -\frac{k_t f'(x_t)}{v_t \delta''(u_t) - k_t(1-\tau)f''(x_t)} < 0,$$

$$\frac{\partial x_t}{\partial \nu} = -\frac{x_t \delta''(u_t)}{v_t \delta''(u_t) - k_t(1-\tau)f''(x_t)} < 0.$$

Recall that $x_t \equiv n_t k_t \equiv u_t k_t / \nu$. An increase in k_t has two opposite effects on x_t: a negative effect on the utilization rate of private capital and a positive effect through an increase in k_t itself. The positive effect dominates over the negative effect on u_t since the elasticity of u_t with respect to k_t is smaller than unity. A rise in either τ or ν decreases x_t because it reduces u_t.

2.2 Dynamic equilibrium and transitional dynamics

This subsection characterizes the macroeconomic equilibrium and its transitional dynamics. We begin our analysis by deriving the dynamic system that represents the dynamic equilibrium. Dynamic equilibrium should satisfy equations (2), (3), (4), (5) and the transversality condition. Let $c_t := C_t / K_t$ as the ratio of private consumption to private capital stock and recall that $u_t = u(k_t, \nu, \tau)$ and $x_t = x(k_t, \nu, \tau)$ are derived from (4). Therefore, the dynamic equilibrium is described by $u_t = u(k_t, \nu, \tau)$, $x_t = x(k_t, \nu, \tau)$ and the following equations:

$$\frac{\dot{c}_t}{c_t} = \frac{(1-\tau)u_t f'(x_t) - \rho - \delta(u_t)}{\theta} - \frac{(1-\tau)u_t f(x_t)}{x_t} + c_t + \delta(u_t), \tag{6}$$

$$\frac{\dot{k}_t}{k_t} = \frac{(1-\tau)u_t f(x_t)}{x_t} - c_t - \delta(u_t) - \tau\nu f(x_t) + \eta(\nu), \tag{7}$$

We define the stationary equilibrium as a dynamic equilibrium such that $\dot{c}_t = \dot{k}_t = 0$ and an asterisk denotes an economic variable's stationary equilibrium value. A stationary equilibrium is a state in which the economy is on a balanced growth path. Therefore, regarding the existence, uniqueness, and stability of such a stationary equilibrium, we establish the following proposition:

Proposition 1. (i) *There exists a unique stable stationary equilibrium with both positive consumption and equilibrium growth rate if*

$$0 < \frac{u^*\delta'(u^*) - \delta(u^*) - \rho}{\theta} < \frac{(1-\tau)u^*f(x^*) - x^*\delta(u^*)}{x^*}.$$

(ii) *Then, the equilibrium growth rate is given as*

$$\gamma^* = \frac{(1-\tau)u^*f'(x^*) - \delta(u^*) - \rho}{\theta} = \frac{u^*\delta'(u^*) - \delta(u^*) - \rho}{\theta},$$

which monotonically increases in the equilibrium utilization rate of private capital.

Proof. See Appendix B.

The inequality in the former half of Proposition 1 is a condition for positive consumption and a positive equilibrium growth rate. The result of the latter half of Proposition 1 is explained as follows: a rise in u^* affects the equilibrium growth rate through both a direct and indirect effect on the net marginal product of private capital service as well as the effect on private capital's depreciation rate. By (4), the effect on private capital's depreciation rate offsets the direct effect on the net marginal product of private capital service. Only the indirect effect on the net marginal product of private capital service remains. This indirect effect depends on the impact of a rise in u^* on the marginal cost of private capital utilization, which is a positive sign. Therefore, the equilibrium growth rate monotonically increases in the equilibrium utilization rate of private capital.

We now consider the transitional dynamics. Solving the linearized system of

(6) and (7) around the stationary equilibrium, we obtain the following lemma:

Lemma 1. *The general solution of a linearized system composed of* (6) *and* (7) *are*

$$c_t = c^* + (k_0 - k^*)\kappa e^{\lambda t} \tag{8}$$

$$k_t = k^* + (k_0 - k^*)e^{\lambda t} \tag{9}$$

where κ is a constant. Note that the following relation holds:

$$\kappa \lesseqgtr 0 \Leftrightarrow \theta \gtreqless \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\phi\psi}{\phi+\psi}\right) \tag{10}$$

where $\psi := u\delta''(u)/\delta'(u)$.

Proof. See Appendix C.

Regarding the dynamics of u_t and x_t, recall equation (4) and $x_t = x(k_t, \nu, \tau)$. The dynamics of x_t depend on k_t and the dynamics of u_t take the opposite of the dynamics of x_t. Therefore, by Lemma 1, the transitional dynamics of this economy can be summarized as follows:

Proposition 2. (i) *Let $\kappa < 0$. Starting from an economy where k_0 is smaller (larger) than its stationary level k^*, both the ratio of private to public capital stock and ratio of private to public capital service increase (decrease) for $t \in (0, \infty)$. Therefore, both the ratio of consumption to private capital stock and the equilibrium utilization rate of private capital decrease (increase) for $t \in (0, \infty)$.* (ii) *Let $\kappa > 0$. Starting from an economy where k_0 is smaller (larger) than k^*, then the ratio of private to public capital stock, ratio of private to public capital service and ratio of consumption to private capital stock increases (decreases) for $t \in (0, \infty)$. Then, the equilibrium utilization rate of private capital decreases (increases) for $t \in (0, \infty)$.*

Figure 1 illustrates the transitional dynamics and explains Proposition 2. The k-nullcline has a downward slope in the k-c plane. Figure 1(a) depicts the case where $\kappa < 0$. The c-nullcline has a downward slope in the k-c plane. The stable trajectory forms a downward curve along the c-nullcline. When $k_0 < k^*$, k_t gradually increase and c_t gradually decrease along the downward stable trajectory. Figure 1(b) depicts the case where $\kappa > 0$. The c-nullcline has an upward slope in the k-c plane. Then, the stable trajectory becomes an upward curve. When $k_0 < k^*$, c_t and k_t gradually increase along the upward stable trajectory.
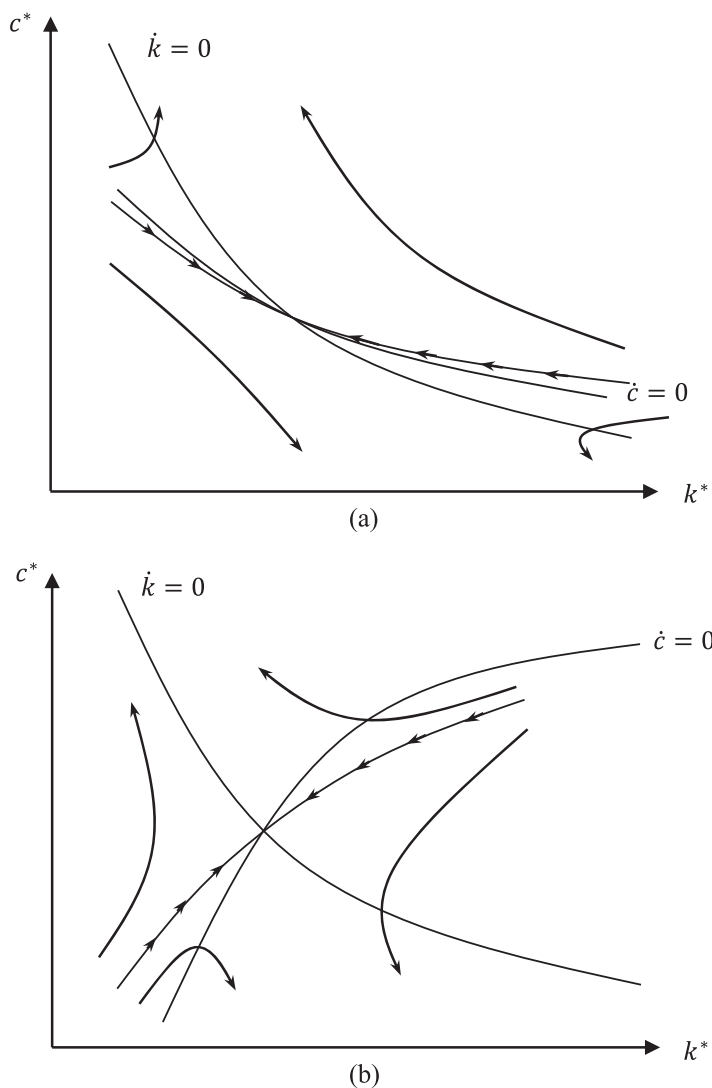

Figure 1.　Phase diagram

3 Macroeconomic effects of income tax

3.1 Long-run effects of income tax

We now consider the long-run effect of a change in τ on economic variables. Note that income tax rate τ is same as the ratio of public investment to total output (i.e. the level of public investment). From the total differentiation of the dynamic system when $\dot{c} = \dot{k} = 0$, we obtain

$$\frac{\partial x^*}{\partial \tau} = \frac{u^* f'(x^*) + \theta \nu f(x^*)}{(1-\tau) u^* f''(x^*) - \theta \tau \nu f'(x^*)} < 0. \tag{11}$$

Using equations (4) and (11), we have

$$\frac{\partial u^*}{\partial \tau} = \left[(1-\tau) f''(x^*) \frac{\partial x^*}{\partial \tau} - f'(x^*) \right] \frac{1}{\delta''(u^*)} \gtreqless 0. \tag{12}$$

Equations (11) and (12) are explained using Figure 2. On the balanced growth path, a rise in τ increases the public capital's growth rate and decreases private consumption's growth rate; the $\dot{G}/G$ curve shifts upward and the $\dot{C}/C$ curve shifts downward in the figure. Accordingly, the intersecting point E' moves to the new point E''. To balance the two growth rates, the ratio of private to public capital service decreases in response to a rise in τ; the value of x^* changes from x^*_{old} to x^*_{new}.

Simultaneously, changes in τ and x^* affects the equilibrium utilization rate of private capital u^* through equation (4). A rise in τ reduces the marginal product of private capital service; the curve represented in equation (4) shifts upward in the figure. Furthermore, a decrease in x^* increases or decreases u^* along the new locus of the curve in (4), although Figure 2 illustrates the former cases: the value of u^* changes from u^*_{old} to u^*_{new}. The U-shape of the dotted curve in the x^*-u^* plane implies the locus of the τ-u^* relation. The break-even point of (12) is the bottom of the U-shaped curve in the figure.

Partial differentiation of equilibrium growth rate yields

$$\frac{\partial \gamma^*}{\partial \tau} = \left[(1-\tau)\frac{\partial u^*}{\partial \tau}f'(x^*) + (1-\tau)u^*f''(x^*)\frac{\partial x^*}{\partial \tau} - u^*f'(x^*) - \frac{\partial u^*}{\partial \tau}\delta'(u^*) \right] \theta^{-1}$$

$$= \left[(1-\tau)f''(x^*)\frac{\partial x^*}{\partial \tau} - f'(x^*) \right] \frac{u^*}{\theta} \gtreqless 0. \tag{13}$$

A rise in τ affects the equilibrium growth rate through its effects on u^*, x^* and τ itself. By (4), the effects of a rise in τ on u^* vanish because a change in u^* affects
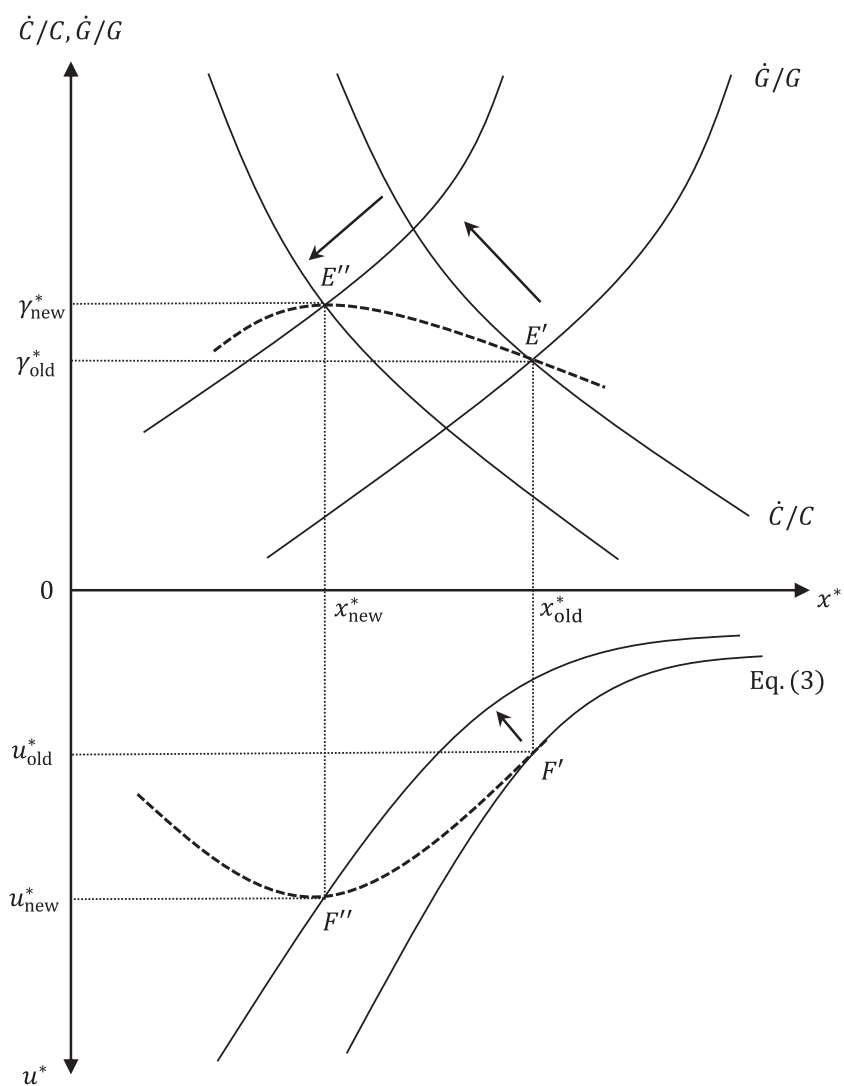

Figure 2.　Comparative statics（a rise in τ）

the depreciation rate of private capital such that the effect on the net marginal product of private capital service is canceled. Therefore, the effects of a rise in τ and a change in x^* on the net marginal product of private capital service remain.

Figure 2 also provides the explanation of the effect on the equilibrium growth rate. The migration lengths of the loci of $\dot{G}/G$ and $\dot{C}/C$ in response to a rise in τ are important for determining the qualitative effect of such a rise in τ on the equilibrium growth rate; Figure 2 illustrates the case where a rise in τ increases the equilibrium growth rate. Accordingly, the relation between the equilibrium growth rate and τ becomes the dotted inverted-U curve passing through E' and E''. The break-even point of (13) is the top of the inverted-U curve in the figure.

Under the optimal utilization rate of private capital, equation (13) manifests a similar form to those found in previous studies (e.g. Futagami et al. 1993; Yakita 2004). However, it differs from previous studies in the fact that the ratio of private to public capital service depends on the marginal cost of private capital utilization and that private and public capital might not be operating at full-capacity. Furthermore, equations (12) and (13) show

$$\operatorname{sgn}\frac{\partial\gamma^*}{\partial\tau} = \operatorname{sgn}\frac{\partial u^*}{\partial\tau}.$$

The above equation implies that maximizing the equilibrium utilization rate of private capital is equivalent to maximizing the equilibrium growth rate. The break-even points of (12) and (13) are same and corresponds to the top and bottom of two dotted curves in Figure 2.

Regarding the growth-maximizing tax rate, we establish the following proposition:

Proposition 3. *Suppose that $\partial^2\gamma^*/\partial\tau^2 < 0$ holds. There exists an income tax rate such that it maximizes both the equilibrium growth rate and the utilization rate of private capital.*

Proof. Let $\partial^2\gamma^*/\partial\tau^2 < 0$. Then, $\partial\gamma^*/\partial\tau$ is decreasing with respect to τ. Considering the limit of (13), we obtain

$$\lim_{\tau \to 0} \frac{\partial \gamma^*}{\partial \tau} = \frac{\theta \nu f(x^*)}{u^*} > 0 \text{ and } \lim_{\tau \to 1} \frac{\partial \gamma^*}{\partial \tau} = -f'(x^*) < 0.$$

These results show that the income tax rate $\bar{\tau}$ such as $\partial \gamma^*/\partial \tau = 0$ is in $(0,1)$ and its tax rate is unique under the above assumption.

Using equations ⑾ and ⒀ with $\partial \gamma^*/\partial \tau = 0$, the growth-maximizing tax rate is given by

$$\bar{\tau} = \frac{\phi(x^*)}{1 - \alpha(x^*) + \phi(x^*)} = \frac{\alpha(x^*)}{[1 - \alpha(x^*)]\epsilon(x^*) + \alpha(x^*)} \tag{14}$$

The second-order output elasticity of public capital service is key to determining the growth-maximizing tax rate. If the second-order output elasticity of public capital service is positive (negative), the growth-maximizing tax rate is larger (smaller) than the output elasticity of public capital service.

Some specific production function forms might give a simple version of the growth-maximizing tax rate. For example, we consider the CES production function of

$$Y_t = A \left[(1 - \beta) Z_t^{-\sigma} + \beta P_t^{-\sigma} \right]^{-1/\sigma}, \tag{15}$$

where $A > 0$, $0 < \beta < 1$, and $\sigma > -1$ are all constants.⑽ Note that the elasticity of substitution becomes $\epsilon = 1/(1+\sigma)$. If the production function is that given in equation ⒂, equation ⒁ can be reduced to

$$\bar{\tau} = \frac{(1 + \sigma)\beta}{(1 - \beta)(1/x^*)^\sigma + (1 + \sigma)\beta}. \tag{16}$$

In the special case where $\epsilon = 1$, we obtain $\bar{\tau} = \beta$.⑾

To characterize the relationship between the public investment, capacity utilization and economic growth, we should investigate the interaction between the marginal

⑽ If the production function takes the form of ⒂, we have $\alpha(x_t) = \beta/[(1 - \beta)x_t^{-\sigma} + \beta]$ and $\phi(x_t) = (1 + \sigma)\beta/[(1 - \beta)x_t^{-\sigma} + \beta]$.

⑾ The case where $\epsilon = 1$ corresponds to the Cobb-Douglas production function.

cost of private capital utilization, public investment and economic variables.　The following lemma describes the effect of a change in the marginal cost of private capital utilization on u^*, x^* and γ^*:

Lemma 2.　*A rise in the marginal cost of private capital utilization reduces the ratio of private capital service to public capital service and raises both the equilibrium utilization rate of private capital and equilibrium growth rate.*

Proof.　Total differentiation of the dynamic system evaluated at the stationary equilibrium and equation (4) provide

$$\frac{\partial x^*}{\partial \delta'} = \frac{1}{(1-\tau)f''(x^*)} < 0, \tag{17}$$

$$\frac{\partial u^*}{\partial \delta'} = \frac{\partial u^*}{\partial x^*}\frac{\partial x^*}{\partial \delta'} = \frac{1}{\delta''(u^*)} > 0,$$

$$\frac{\partial \gamma^*}{\partial \delta'} = \frac{u^*}{\theta} > 0.$$

Lemma 2 is explained by Figure 3, which provides graphical representation of equation (4).　Point E' corresponds to the initial state that satisfies equation (4). Since the marginal product of private capital services is decreasing function with respect to the ratio of private to public capital service, it is depicted as a downward curve.　The marginal cost of private capital utilization is a horizontal line.　A rise in the marginal cost of private capital utilization shifts the horizontal curve upward.　The intersecting point E' moves to new point E''.　Consequently, the ratio of private to public capital service decreases in response to an increase in the marginal cost of private capital utilization.

Private capital's equilibrium utilization rate rise in response to an increase in the marginal cost of its utilization.　This occurs because this marginal utilization cost is an increasing function with respect to the private capital utilization rate, as shown in Figure 3.　Finally, a rise in the marginal cost of private capital utilization raises the equilibrium growth rate through increasing the equilibrium utilization

rate of private capital.

Using Lemma 2 and ⑯, we obtain the following result:

Proposition 4.　*Suppose that the production function takes the form of equation ⑮.　The growth-maximizing tax rate is increasing（decreasing）with respect to the marginal cost of private capital utilization if the elasticity of substitution in the production function is larger（smaller）than unity.*
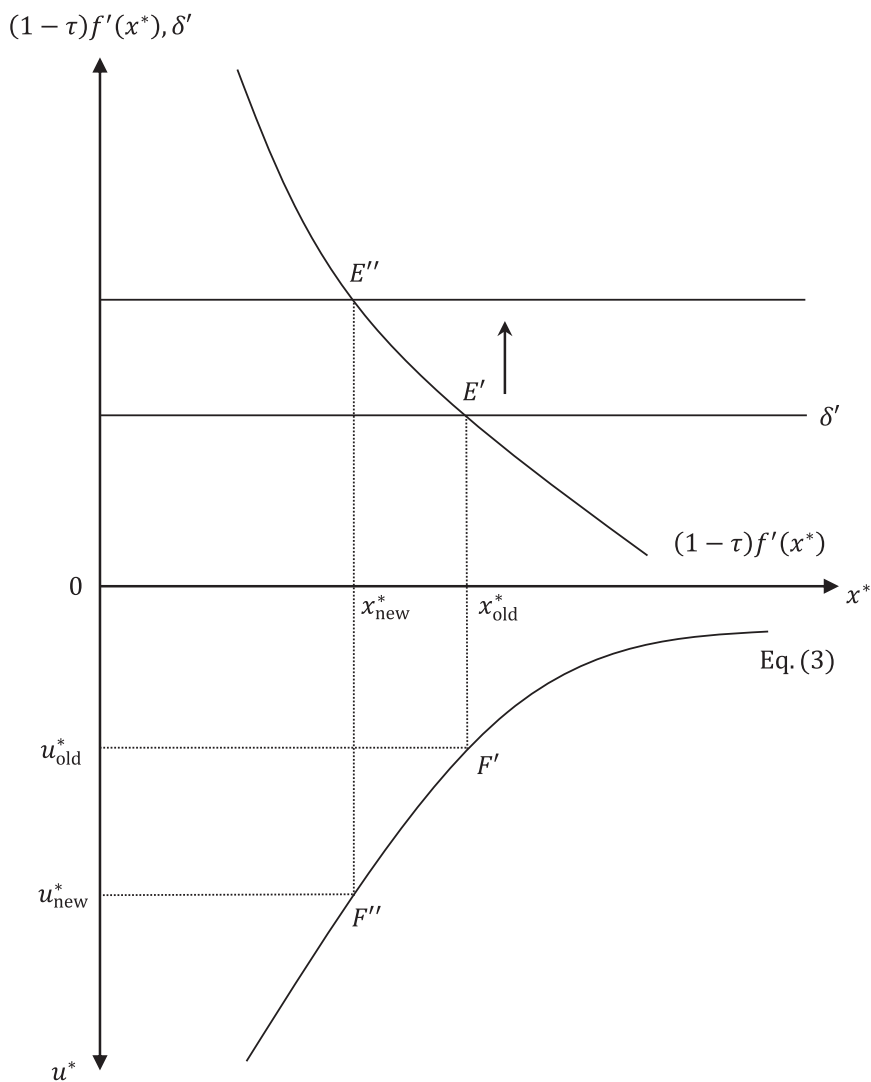

Figure 3.　Comparative statics（a rise in δ'）

Proof. Taking into account equation (17), the partial differentiation of (16) with respect to δ' yield

$$\frac{\partial \bar{\tau}}{\partial \delta'} = \frac{\beta \sigma (1-\beta)(1+\sigma)(1/x^*)^{1+\sigma} \partial x^*/\partial \delta'}{[(1-\beta)(1/x^*)^{\sigma} + (1+\sigma)\beta]^2 - \sigma(1-\beta)(1/x^*)^{1+\sigma} \partial x^*/\partial \bar{\tau}} \gtreqless 0$$

$$\Leftrightarrow \sigma \lesseqgtr 0 \Leftrightarrow \epsilon \gtreqless 1.$$

Equation (14) shows that the growth-maximizing tax rate depends on the ratio of private capital service to public capital service. Accordingly, capacity utilization affects the growth-maximizing tax rate through the input substitutability or complementarity. Specifically equation (16) implies that the growth-maximizing tax rate is increasing at the ratio of private capital service to public capital service if $\sigma > 0$, i.e., $\epsilon < 1$, and the growth-maximizing tax rate is decreasing at the ratio of private capital service to public capital service if $\sigma < 0$, i.e., $\epsilon > 1$. As shown in Lemma 2, an increase in the marginal cost of private capital utilization decreases the ratio of private capital service to public capital service. Therefore, an increase in the marginal cost of private capital utilization raises (reduces) the growth-maximizing tax rate through a decrease in the ratio of private capital service to public capital service if $\epsilon > 1$ ($\epsilon < 1$).

3.2 Dynamic effects of income tax

We now characterize the dynamic responses shown by economic variables with respect to a change in τ. Suppose that the economy initially exists in a stationary equilibrium, and an unexpected increase in tax rate occurs at $t = 0$. Note that a change in τ does not affect the initial ratio of private to public capital stock k_0, whereas the initial ratio of private to public capital service x_0 is affected by such a change. In other words, a change in τ has no instantaneous effect on the ratio of private to public capital stock. Using (9), the dynamic effect of a change in τ on k_t is $\partial k_t/\partial \tau = (1 - e^{\lambda t}) \partial x^*/\partial \tau \le 0$ for $t \ge 0$.

Applying the method of comparative dynamics presented by Judd (1982, 1985), we obtain the following result regarding the initial effects on the three jumpable

economic variables, c_t, u_t, and x_t:

Lemma 3.　*Suppose that $\theta \geq 1$ holds.　The effects of public investment on initial private consumption, the utilization rate of private capital, and ratio of private to public capital are*

$$\operatorname{sgn}\frac{\partial c_0}{\partial \tau}\bigg|_{\frac{\partial \gamma}{\partial \tau}=0} = \operatorname{sgn}\frac{\partial C_0}{\partial \tau}\bigg|_{\frac{\partial \gamma}{\partial \tau}=0} \gtreqless 0 \Leftrightarrow \frac{\psi}{\phi} \gtreqless \frac{\mu\theta}{(1-\alpha-\mu)\theta+(c^*-\mu)k^*}, \tag{18}$$

$$\frac{\tau}{u_0}\frac{\partial u_0}{\partial \tau} = -\frac{\tau}{(1-\tau)(\phi+\psi)} < 0, \tag{19}$$

$$\frac{\tau}{x_0}\frac{\partial x_0}{\partial \tau} = \frac{\tau}{u_0}\frac{\partial u_0}{\partial \tau} < 0. \tag{20}$$

Proof.　See Appendix D.

As shown in (18), ψ and ϕ are important in determining the sign of the instantaneous effect of a rise in τ on private consumption when $\partial \gamma^*/\partial \tau = 0$.　Note that a rise in τ has a positive instantaneous partial effect on income and a negative substitution effect.　If ψ is sufficiently small, the instantaneous partial effect on income through a rise in the equilibrium utilization rate of private capital is also sufficiently small.　As a result, the income effect of a rise in τ is totally negative and then a rise in τ has a negative impact on private consumption.　However, if ψ is sufficiently large, the instantaneous partial effect on income through a rise in u_0 is also sufficiently large.　A rise in τ has an entirely positive income effect, and if it dominates over a negative substitution effect, it also increases initial private consumption.

The size of ϕ affects the impact of ψ: a higher ϕ reduces the impact of ψ by (18).　A sufficient large (small) ϕ is sufficiently large corresponds to the case where ψ is sufficiently small (large).　In other words, a rise in τ decreases (increases) instantaneous consumption where the elasticity of input substitution is sufficiently small (large) or where, equivalently, the second-order elasticity φ is negative (positive) and sufficiently large (small).

According to (4), a rise in τ at $t = 0$ decreases the net marginal product of private capital service and also decreases the marginal cost of private capital utilization.

Consequently, a rise in τ reduces the equilibrium utilization rate of private capital at $t = 0$. Since k_0 is independent of τ, a rise in τ decreases x_0 because it reduces u_0. Recall $\partial k_t / \partial \tau = (1 - e^{\lambda t}) \partial x^* / \partial \tau \leq 0$ for $t \geq 0$. Following the instantaneous effects of a rise in τ on u_0 and x_0, u_t and x_t gradually change to their stationary values.

Using Lemma 3, the partial differentiation of (2), (3) and (5) with respect to τ at $t = 0$ yields

$$\frac{\partial}{\partial \tau}\left(\frac{\dot{C}_0}{C_0}\right) = -\frac{\psi}{\phi + \psi}\frac{u_0 f'(x_0)}{\theta} < 0, \tag{21}$$

$$\frac{\partial}{\partial \tau}\left(\frac{\dot{K}_0}{K_0}\right) = -\frac{u_0 f(x_0)}{x_0} - \frac{\partial c_0}{\partial \tau}, \tag{22}$$

and

$$\frac{\partial}{\partial \tau}\left(\frac{\dot{G}_0}{G_0}\right) = \nu f(x_0)\left[1 + (1 - \alpha_0)\frac{\tau}{u_0}\frac{\partial u_0}{\partial \tau}\right] \gtreqless 0 \Leftrightarrow \tau \lesseqgtr \frac{\phi + \psi}{1 - \alpha + \phi + \psi}, \tag{23}$$

where $\alpha_0 := \alpha(x_0)$. As shown in Lemma 3, a rise in τ has negative instantaneous effects on the equilibrium utilization rate of private capital and ratio of private to public capital service.

The instantaneous effect on the private consumption growth rate, (21), depends on the effects on the net marginal product of private capital service through both a tax burden effect and change in x_0. As these two effects are negative, a rise in τ decreases $\dot{C}_0 / C_0$. As shown in equation (22), the instantaneous effect on the growth rate of private capital stock is affected by two instantaneous effects on disposal income and private consumption. If $\partial c_0 / \partial \tau > 0$, the total effect on $\dot{K}_0 / K_0$ is negative. However, if $\partial c_0 / \partial \tau < 0$, the total effect is ambiguous. In the latter case, it is appropriate to assume that the marginal change of private consumption to disposal income is less than unity.[12] The instantaneous effect on the growth rate of public capital stock, (23), depends on a positive direct effect on public investment and a negative

---

[12] In other words, a decrement amount of C_0 is not greater than a decrement amount of $(1-\tau)Y_0$

indirect effect on public investment through a change in u_0. For a small τ, the direct effect dominates over the indirect effect. These results are illustrated in Figure 4(a).

The above results evaluated at $\partial\gamma^*/\partial\tau = 0$ are summarized as follows:

Proposition 5. *Suppose that the income tax rate is equal to the growth-maximizing tax rate at $t = 0$. An unexpected rise in the income tax rate leads the growth rates of both private consumption and private capital stock undershoot the equilibrium growth rate; meanwhile, the growth rate of public capital stock overshoots（undershoots）the equilibrium growth rate if an increment of income tax rate is sufficiently small（large）.*

Regarding the effect of a change in τ on the time rate of change of the equilibirum private capital utilization rate, we derive

$$\frac{\partial}{\partial\tau}\left(\frac{\dot{u}_0}{u_0}\right) = \left[\frac{\partial}{\partial\tau}\left(\frac{\dot{G}_0}{G_0}\right) - \frac{\partial}{\partial\tau}\left(\frac{\dot{K}_0}{K_0}\right)\right]\frac{\psi}{\phi+\psi} > 0 \tag{24}$$

from equation (4) after some manipulations.[13] The result derived from equation (24) illustrates Figure 4(b). By Lemma 3, a rise in τ has a negative instantaneous effect on the equilibrium utilization rate of private capital. However, after this initial effect, this utilization rate gradually increases to converge on its new stationary value through a gradual decrease in x_t. This effect can be reduced to the term $[\partial(\dot{G}_0/G_0)/\partial\tau - \partial(\dot{K}_0/K_0)/\partial\tau] > 0$.

The logarithmic differentiation of the production function with respect to τ at $t = 0$ show that the instantaneous effect of a rise in τ on economic growth rate is

$$\frac{\partial}{\partial\tau}\left(\frac{\dot{Y}_0}{Y_0}\right) = (1-\alpha)\left[\frac{\partial}{\partial\tau}\left(\frac{\dot{u}_0}{u_0}\right) + \frac{\partial}{\partial\tau}\left(\frac{\dot{K}_0}{K_0}\right)\right] + \alpha\frac{\partial}{\partial\tau}\left(\frac{\dot{G}_0}{G_0}\right) \tag{25}$$

---

[13] Logarithmic derivation of (4) and $x_t = n_t k_t$ are $\psi\dot{u}_t/u_t = -\phi\dot{x}_t/x_t$ and $\dot{x}_t/x_t = \dot{u}_t/u_t + \dot{k}_t/k_t$ respectively. Using these equations and (7) with $t = 0$, we obtain $\dot{u}_0/u_0 = \psi[\dot{G}_0/G_0 - \dot{K}_0/K_0]/(\phi+\psi)$. The partial differentiation of $\dot{u}_0/u_0$ with respect to τ provides $\partial(\dot{u}_0/u_0)/\partial\tau$.
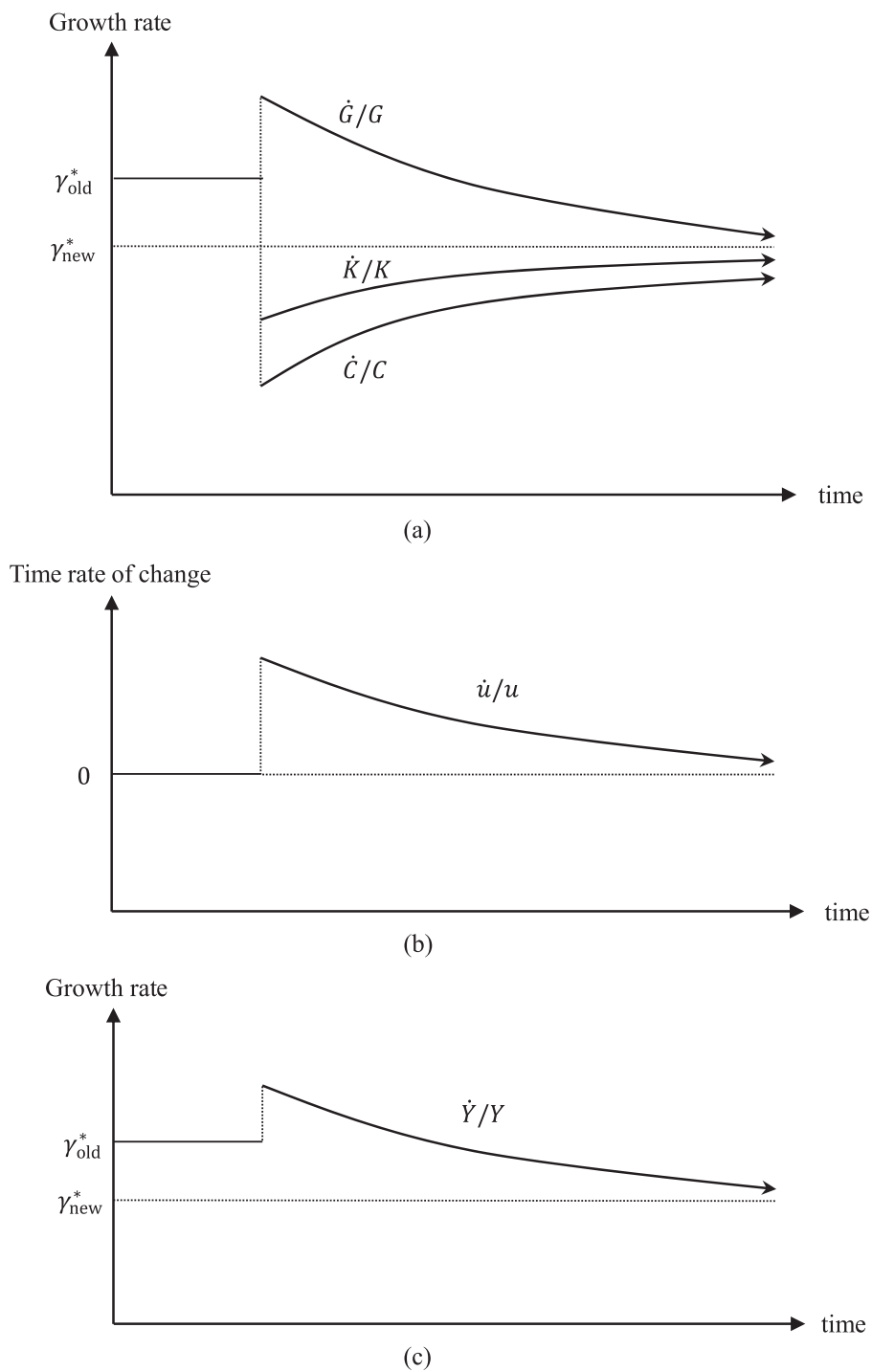
Growth rate

$\dot{G}/G$

γ^*_{old}

γ^*_{new}

$\dot{K}/K$

$\dot{C}/C$

time

(a)

Time rate of change

$\dot{u}/u$

0

time

(b)

Growth rate

$\dot{Y}/Y$

γ^*_{old}

γ^*_{new}

time

(c)

Figure 4. Transitional dynamics（a rise in τ）

$$= \frac{\alpha\phi + \psi}{\phi + \psi} \frac{\partial}{\partial \tau} \left(\frac{\dot{G}_0}{G_0} \right) + \frac{(1-\alpha)\phi}{\phi + \psi} \frac{\partial}{\partial \tau} \left(\frac{\dot{K}_0}{K_0} \right).$$

The relative size of the positive effect on $\dot{G}_0/G_0$ to the negative effect on $\dot{K}_0/K_0$ is important to determine the sign of equation ㉕. This occurs because the sum of the instantaneous effects of $\dot{u}_0/u_0$, $\dot{K}_0/K_0$ and $\dot{G}_0/G_0$, i.e. the right hand side of the first line in equation ㉕, is reduced to the second line in equation ㉕. When the size of a positive effect on $\dot{G}_0/G_0$ is sufficiently larger than that of a negative effect on $\dot{K}_0/K_0$, the economic growth rate at $t = 0$ increases in response to a rise in τ. This case is illustrated in Figure 4(c).

These results are summarized as the following proposition:

Proposition 6. *Suppose that the income tax rate is equal to the growth-maximizing tax rate at $t = 0$ and an increment of income tax rate is sufficiently small. In response to an unexpected rise in the income tax rate, the time rate of change for the equilibrium private capital utilization rate overshoots the equilibrium time rate of change that equals zero. Meanwhile, the economic growth rate overshoots (undershoots) the equilibrium growth rate if the overshoot of the public capital stock growth rate is sufficiently larger (smaller) than the undershoot of the private capital stock growth rate.*

Finally, we derive the welfare effect of a change in τ. On the balanced-growth path, partial differentiation of the indirect utility function with respect to τ yields

$$\frac{\partial W_0}{\partial \tau} = C_0^{1-\theta} \int_0^\infty \left[\frac{1}{C_0} \frac{\partial C_0}{\partial \tau} + \int_0^t \frac{\partial}{\partial \tau} \left(\frac{\dot{C}_s}{C_s} \right) ds \right] e^{-\Gamma t} dt, \qquad ㉖$$

where $\Gamma := \rho + (\theta - 1)\gamma^*$ is assumed to be a positive constant. The first term on the right hand side of ㉖ is the instantaneous effect of a change in income tax rate on private consumption, and the second term is the effect of a change in income tax rate on the private consumption growth rate. Unlike the models with full-capacity operation, these two processes include the an income tax rate change on the utilization of private capital, which is a newly added effect taking into account endogenous choice of capacity utilization.

Regarding the welfare-maximizing tax rate, using equation ㉖, we obtain the following proposition:

Proposition 7. *The welfare-maximizing tax rate is less than the growth-maximizing tax rate if ψ/ϕ is sufficiently small. However, if ψ/ϕ is sufficiently large, the welfaremaximizing tax rate might be greater than the growth-maximizing tax rate.*

Proof. Evaluating ㉖ at the growth-maximizing tax rate, we obtain

$$
\operatorname{sgn} \left. \frac{\partial W_0}{\partial \tau} \right|_{\frac{\partial \gamma}{\partial \tau}=0} = \operatorname{sgn} \left[\left. \frac{1}{C_0} \frac{\partial C_0}{\partial \tau} \right|_{\frac{\partial \gamma}{\partial \tau}=0} + \int_0^t \left. \frac{\partial}{\partial \tau} \left(\frac{\dot{C}_s}{C_s} \right) \right|_{\frac{\partial \gamma}{\partial \tau}=0} ds \right], \tag{㉗}
$$

where

$$
\int_0^t \left. \frac{\partial}{\partial \tau} \left(\frac{\dot{C}_s}{C_s} \right) \right|_{\frac{\partial \gamma}{\partial \tau}=0} ds = - \frac{\psi}{\phi + \psi} \frac{u^* f'(x^*)}{\theta} \frac{e^{\lambda t} - 1}{\lambda} < 0.
$$

If ψ/ϕ is sufficiently small, the first term in the right hand side of ㉗ is negative by Lemma 3 and the sum of the terms in the right hand side of ㉗ is also negative. Therefore, we obtain $\partial W_0/\partial \tau < 0$ at $\tau = \bar{\tau}$. If ψ/ϕ is sufficiently large, the first term in the right hand side of ㉗ is positive by Lemma 3 and the sum of the terms in the right hand side of ㉗ might be positive. Then, $\partial W_0/\partial \tau > 0$ might hold at $\tau = \bar{\tau}$.

Proposition 7 implies that a growth-maximizing government has not only an incentive to over-invest in but also under-invest in public capital stock. As shown in Lemma 3, a rise in τ incurs the possibility of increasing disposal income through a rise in the utilization rate of private capital, which might then instantaneously increase private consumption. Households benefit from increased disposal income for a short while although a rise in τ has a negative effect on the growth rate of private consumption. A higher cost for the equilibrium utilization rate of private capital boosts the possibility of a rise in τ having an increasing positive effect on the welfare. In the case where private capital utilization has a low cost, the result

of Proposition 7 bucks the model without capacity utilization.

4　Macroeconomic effects of public capital utilization

4.1　Long-run effects of public capital utilization

We now consider the long-run economic effect of a change in ν. The total differentiation of a dynamic system when $\dot{c} = \dot{k} = 0$ provides

$$\frac{\partial x^*}{\partial \nu} = \frac{[\tau f(x^*) - \eta'(\nu)]\theta}{(1-\tau)u^* f''(x^*) - \theta\tau\nu f'(x^*)} \gtreqless 0 \Leftrightarrow \eta'(\nu) \gtreqless \tau f(x^*). \tag{28}$$

Using (4) and (28), we obtain

$$\frac{\partial u^*}{\partial \nu} = (1-\tau)\frac{f''(x^*)}{\delta''(u^*)}\frac{\partial x^*}{\partial \nu} \gtreqless 0 \Leftrightarrow \eta'(\nu) \lesseqgtr \tau f(x^*). \tag{29}$$

Equations (28) and (29) are interpreted using Figure 5. In response to a rise in ν, the $\dot{G}/G$ curve moves upward (downward) if ν is sufficiently small (large). Then, the $\dot{C}/C$ curve remains static. When ν is sufficiently small, the stationary equilibrium point E' moves to new point E''. Therefore, a rise in ν increases x^*. When ν is sufficiently small, the opposite mechanism operates to equalize the growth rates of private consumption and public capital stock. The effect of a rise in ν on u^* is explained by a shift of the curve representing equation (4). A rise in τ leads the equilibrium utilization rate of private capital to increase along the curved path: $F' \to F''$.

The partial differentiation of the equilibrium growth rate with respect to ν yeilds

$$\frac{\partial \gamma^*}{\partial \nu} = \left[(1-\tau)\frac{\partial u^*}{\partial \nu}f'(x^*) + (1-\tau)u^* f''(x^*)\frac{\partial x^*}{\partial \nu} - \frac{\partial u^*}{\partial \nu}\delta'(u^*)\right]\theta^{-1}$$

$$= (1-\tau)\frac{u^* f''(x^*)}{\theta}\frac{\partial x^*}{\partial \nu} \gtreqless 0 \Leftrightarrow \eta'(\nu) \lesseqgtr \tau f(x^*). \tag{30}$$

Equation (30) implies that a rise in ν affects the equilibrium growth rate through

its effects on u^* and x^*.　Following equation (4), the effect of a change in u^* on the private capital depreciation rate offsets its effect on the net marginal product of private capital service.　　Therefore, there remain the effects of a change in x^* stemming from a rise in ν remain in operation upon the net marginal product of private capital service.　Equations (29) and (30) derive

$$\mathrm{sgn}\frac{\partial\gamma^*}{\partial\nu} = \mathrm{sgn}\frac{\partial u^*}{\partial\nu}.$$
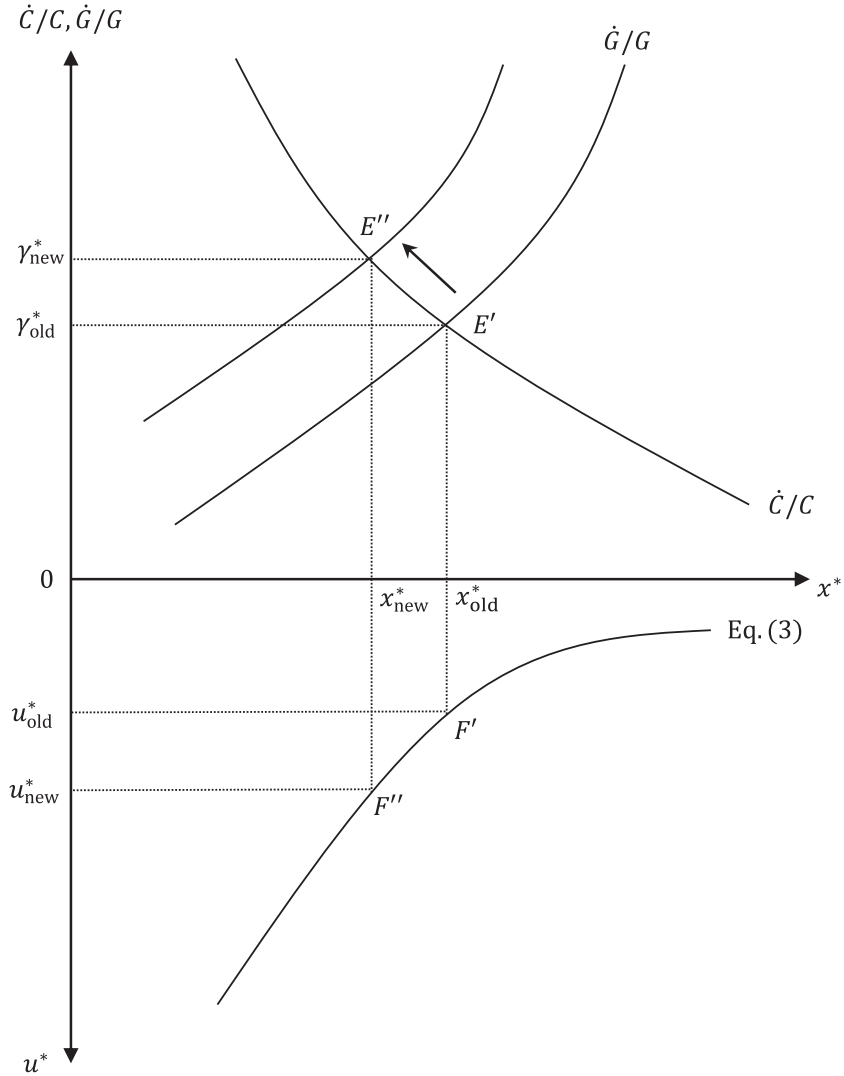

Figure 5.　Comparative statics（a rise in ν）

This equation shows that maximizing the equilibrium utilization rate of private capital is equivalent to maximizing the equilibrium growth rate.

Regarding the growth-maximizing utilization rate of public capital, we establish the following proposition:

Proposition 8. *Suppose that $\partial^2 x^*/\partial\nu^2 < 0$ holds and $\lim_{\nu\to 1}\eta'(\nu)$ is sufficiently large. There exists a utilization rate of public capital such that one maximizes both the equilibrium growth rate and equilibrium utilization rate of private capital.*

Proof. Since $\partial^2\gamma^*/\partial\nu^2 < 0$, $\partial\gamma^*/\partial\nu$ is decreasing with respect to ν. Taking the limit of ㉚, we obtain

$$\lim_{\nu\to 0}\frac{\partial\gamma^*}{\partial\nu} = (1-\tau)\lim_{\nu\to 0}\left[\frac{f''(x^*)}{\delta''(u^*)}\frac{\theta\tau f(x^*)}{(1-\tau)u^* f''(x^*) - \theta\tau\nu f'(x^*)}\right] > 0$$

$$\lim_{\nu\to 1}\frac{\partial\gamma^*}{\partial\nu} = (1-\tau)\lim_{\nu\to 1}\left[\frac{f''(x^*)}{\delta''(u^*)}\frac{[\tau f(x^*) - \eta'(\nu)]\theta}{(1-\tau)u^* f''(x^*) - \theta\tau\nu f'(x^*)}\right].$$

The sign of $\lim_{\nu\to 1}\partial\gamma^*/\partial\nu$ depends on the sign of $\lim_{\nu\to 1}[\tau f(x^*) - \eta'(\nu)]$. If $\lim_{\nu\to 1}\eta'(\nu)$ is sufficiently large, $\lim_{\nu\to 1}[\tau f(x^*) - \eta'(\nu)]$ is negative. Accordingly, we have $\lim_{\nu\to 1}\partial\gamma^*/\partial\nu < 0$. These results show that the utilization rate of public capital $\bar{\nu}$ such as $\partial\gamma^*/\partial\nu = 0$ is in $(0, 1)$ and its value is uniquely determined.

If the marginal cost of public capital utilization is sufficiently large at full-capacity operation, the growth-maximizing utilization rate of public capital is less than the fullcapacity operation level. However, if the marginal cost of public capital utilization is not so large, full-capacity operation is desirable for maximizing the equilibrium growth rate. According to equation ㉚, the growth-maximizing utilization rate of public capital is necessary to satisfy the requirement that the marginal cost of public capital utilization equals the ratio of public investment to public capital service. Thus, the growth-maximizing utilization rate of public capital depends on the ratio of private to public capital service. This mechanism is also observed in Figure 5.

To characterize the effects of capacity utilization, we consider the relationship between the marginal cost of private capital utilization and the growth-maximizing utilization rate of public capital as well as the relationship between the utilization rate of public capital and the growth-maximizing tax rate. Using Lemma 2 and equation ㉚ with $\partial \gamma^*/\partial \nu = 0$, we obtain the following proposition:

Proposition 9. *Suppose that $\partial^2 x^*/\partial \nu^2 < 0$ holds and $\lim_{\nu \to 1} \eta'(\nu)$ is sufficiently large. The growth-maximizing utilization rate of public capital is decreasing at the marginal cost of private capital utilization.*

Proof. Equation ⑰ and total differentiation of the growth-maximizing condition for ν provide

$$\frac{\partial \bar{\nu}}{\partial \delta'} = \frac{\tau f'(x^*)}{\eta''(\bar{\nu})} \frac{\partial x^*}{\partial \delta'} < 0.$$

Following Lemma 2, a rise in the marginal cost of private capital utilization decreases the ratio of private to public capital service. A decrease in x^* reduces the positive effect of a rise in ν on the public investment and therefore the marginal cost of public capital utilization should be also reduced to balance these two effects. A rise in δ' always reduces the growth-maximizing utilization rate of public capital while also increasing or decreasing the growth-maximizing tax rate according to the elasticity of input substitution ϵ.

Using ⑪, ⑯ and ㉘, the relation between the utilization rate of public capital and the growth-maximizing tax rate is summarized as follows:

Proposition 10. *Suppose that the production function takes the form of equation ⑮. (i) $\epsilon > 1$. A rise in the utilization rate of public capital reduces (raises) the growthmaximizing tax rate if the utilization rate of public capital is smaller (larger) than its growth-maximizing rate. (ii) $\epsilon < 1$. A rise in the utilization rate of public capital raises (reduces) the growth-maximizing tax rate if the utilization rate of public capital is smaller (larger) than its growth-maximizing rate.*

Proof.　Taking equation ㉘ into account, the partial differentiation of ⑯ with respect to ν yield

$$\frac{\partial \bar{\tau}}{\partial \nu} = \frac{\beta\sigma(1-\beta)(1+\sigma)(1/x^*)^{1+\sigma}\partial x^*/\partial \nu}{[(1-\beta)(1/x^*)^\sigma + (1+\sigma)\beta]^2 - \sigma(1-\beta)(1/x^*)^{1+\sigma}\partial x^*/\partial \tau} \gtreqless 0$$

$$\Rightarrow \operatorname{sgn}\frac{\partial \bar{\tau}}{\partial \nu} = \operatorname{sgn}(\bar{\nu} - \nu)\sigma = \operatorname{sgn}(\bar{\nu} - \nu)(1 - \epsilon) \gtreqless 0.$$

The factor of capacity utilization of public capital affects the growth-maximizing tax rate through a change in the ratio of private to public capital service.　Therefore, the elasticity of input substitution is key to determing the impact of a change in x^* upon the growth-maximizing tax rate.　If public capital service is a complement of private capital service ($\epsilon < 1$), then an increase in x^* raises the growth-maximizing tax rate.　By Proposition 8, a utilization rate of public capital exists that maximizes the equilibrium growth rate.　Therefore, both the elasticity of input substitution and capacity utilization of public capital are important in determining the effect of a change in ν upon the growth-maximizing tax rate.

4.2　Dynamic effects of public capital utilization

We now investigate the effect of a change in ν on the transitional paths of economic variables.　As in Section 3, we assume that the economy initially exists in a stationary equilibrium, and that an unexpected increase in tax rate occurs at $t = 0$.　Comparative dynamics analysis provides the following result:

Lemma 4.　*The effects of public capital utilization on initial private consumption and the equilibrium utilization rate of private capital are*

$$\operatorname{sgn}\frac{\partial c_0}{\partial \nu} = \operatorname{sgn}\frac{\partial C_0}{\partial \nu} \gtreqless 0 \Leftrightarrow \kappa \gtreqless 0, \tag{31}$$

$$\frac{\nu}{u_0}\frac{\partial u_0}{\partial \nu} = \frac{\phi}{\phi + \psi} \in (0, 1], \tag{32}$$

$$\frac{\nu}{x_0}\frac{\partial x_0}{\partial \nu} = \frac{\nu}{u_0}\frac{\partial u_0}{\partial \nu} - 1 = -\frac{\psi}{\phi + \psi} \in (-1, 0]. \tag{33}$$

Proof. See Appendix D.

Equation (31) can be explained as follows. Note that κ relates to the gradient of c-nullcline as well as to both the gradient and intercept of the unique stable trajectory in Figure 1. When $\kappa < 0$, a rise in ν leads the c-nullcline to move down in the k^*-c^* plane. Regardless of the response by k-nullcline, the equilibrium trajectory also moves down, and accordingly, the initial private consumption decrease, jumping from its initial trajectory to a new equilibrium trajectory. When $\kappa < 0$, a rise in ν leads the c-nullcline to move upward in the k^*-c^* plane. The equilibrium trajectory also moves upward, and therefore initial private consumption increase, jumping to a new equilibrium trajectory.

Equations (32) and (33) have similar explanations to equations (19) and (20). A rise in ν increases the net marginal product of private capital service at $t = 0$, and therefore should increase the marginal cost of private capital utilization. Thus, a rise in ν raises the equilibrium utilization rate of private capital u_0. In addition, a rise in ν decreases the ratio of private to public capital service x_0 because an increase in u_0 raises the net marginal product of private capital service.

Using Lemma 4 and the partial derivatives of (2), (3) and (5) with respect to ν at $t = 0$, we obtain

$$\frac{\partial}{\partial \nu}\left(\frac{\dot{C}_0}{C_0}\right) = \frac{u_0 \delta''(u_0)}{\theta}\frac{\partial u_0}{\partial \nu} > 0, \tag{34}$$

$$\frac{\partial}{\partial \nu}\left(\frac{\dot{K}_0}{K_0}\right) = \frac{(1-\tau)\alpha n_0 f(x_0)}{x_0} - \frac{\partial c_0}{\partial \nu}, \tag{35}$$

and

$$\frac{\partial}{\partial \nu}\left(\frac{\dot{G}_0}{G_0}\right) = \tau f(x_0)\left[1 + (1-\alpha)\frac{\nu}{x_0}\frac{\partial x_0}{\partial \nu}\right] - \eta'(\nu) \gtreqless 0 \tag{36}$$

$$\Leftrightarrow \eta'(\nu) \lesseqgtr \frac{(1-\alpha)\phi\tau f(x_0)}{\phi + \psi}.$$

Equation (34) shows that a rise in ν has a positive instantaneous effect on the growth

rate of private consumption.　This occurs because a rise in ν increases the marginal product of private capital service by raising the equilibrium utilization rate of private capital.　　If $\kappa < 0$, the instantaneous effect of a rise in ν on the growth rate of private capital stock is also positive by equations ③① and ③⑤ since a rise in ν has a negative instantaneous effect on private consumption.　　As shown in equation ③⑥, the instantaneous effect produced by a rise in ν upon the growth rate of public capital stock depends on the marginal cost of public capital utilization $\eta'(\nu)$.　For low (high) η', a positive instantaneous effect on public investment is larger (smaller) than the marginal cost of public capital utilization, and therefore, a rise in ν increases (decreases) the initial growth rate of public capital stock.　Figure 6(a) illustrates the case where $\kappa < 0$ and $\partial \gamma^*/\partial \nu = 0$.

The above results are summarized as follows.

Proposition 11.　*Suppose that public capital's actual utilization rate equals its growthmaximizing utilization rate and $\kappa < 0$.　An unexpected rise in the utilization rate of public capital leads the growth rate of private consumption to overshoot the equilibrium growth rate, the growth rate of private capital stock to overshoot the equilibrium growth rate, and the growth rate of public capital stock to undershoot the equilibrium growth rate.*

Similar to the method used to deriving $\partial(\dot{u}_0/u_0)/\partial \tau$ and $\partial(\dot{Y}_0/Y_0)/\partial \tau$ in Section 4, the effect of a change in ν on the time rate of change for the equilibrium utilization rate of private capital and on the economic growth rate are given by

$$\frac{\partial}{\partial \nu}\left(\frac{\dot{u}_0}{u_0}\right) = \left[\frac{\partial}{\partial \nu}\left(\frac{\dot{G}_0}{G_0}\right) - \frac{\partial}{\partial \nu}\left(\frac{\dot{K}_0}{K_0}\right)\right]\frac{\psi}{\phi + \psi} < 0, \tag{37}$$

$$\frac{\partial}{\partial \nu}\left(\frac{\dot{Y}_0}{Y_0}\right) = (1 - \alpha)\left[\frac{\partial}{\partial \nu}\left(\frac{\dot{u}_0}{u_0}\right) + \frac{\partial}{\partial \nu}\left(\frac{\dot{K}_0}{K_0}\right)\right] + \alpha\frac{\partial}{\partial \nu}\left(\frac{\dot{G}_0}{G_0}\right)$$

$$= \frac{\alpha\phi + \psi}{\phi + \psi}\frac{\partial}{\partial \nu}\left(\frac{\dot{G}_0}{G_0}\right) + \frac{(1 - \alpha)\phi}{\phi + \psi}\frac{\partial}{\partial \nu}\left(\frac{\dot{K}_0}{K_0}\right). \tag{38}$$

The transitional effect of a rise in ν on the time rate of change for the equilibrium utilization rate of private capital is illustrated in Figure 6(b).　By equations ②⑨ and
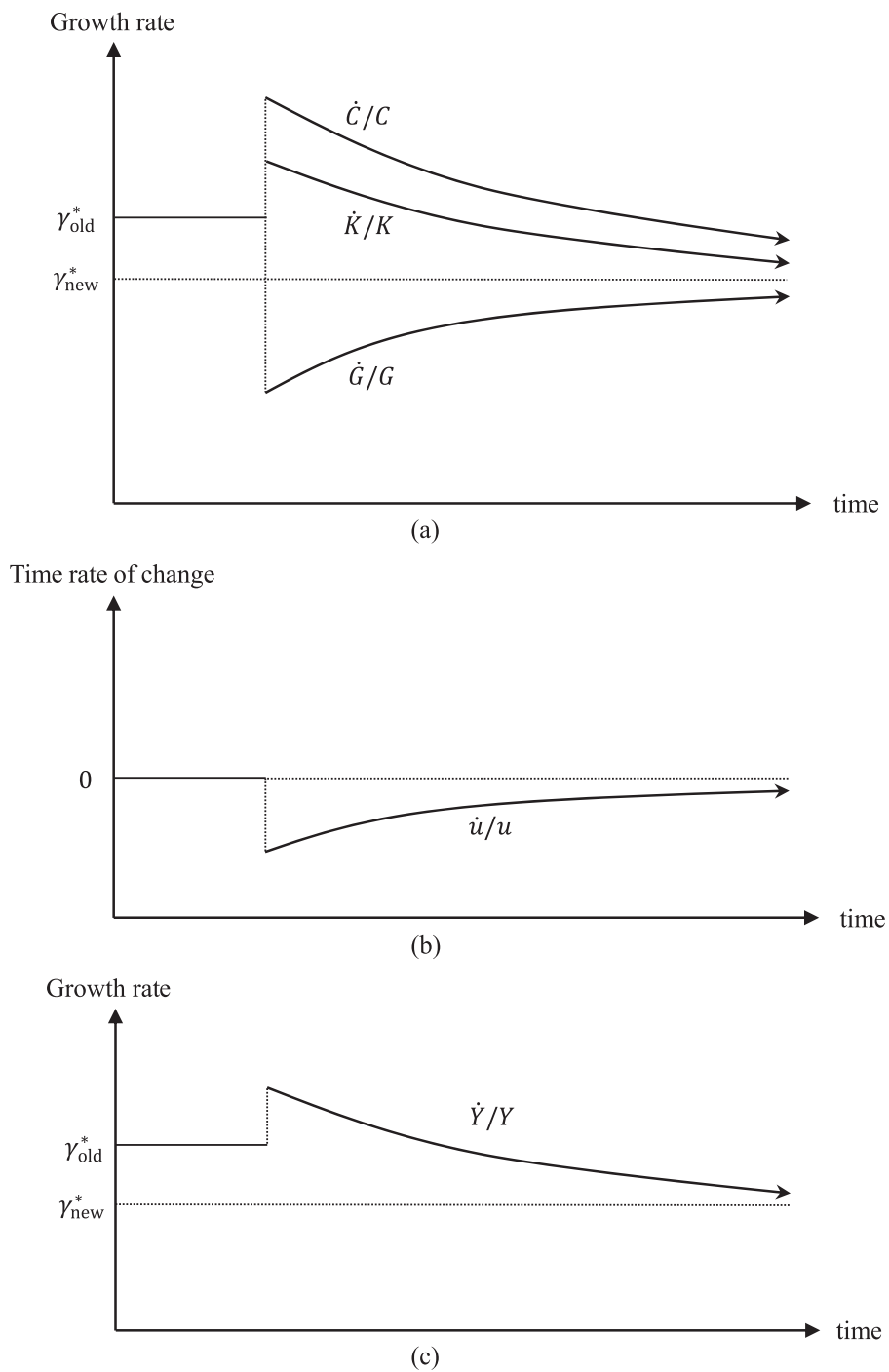
Growth rate

$\dot{C}/C$

$\dot{K}/K$

γ^*_{old}

γ^*_{new}

$\dot{G}/G$

time

(a)

Time rate of change

0

$\dot{u}/u$

time

(b)

Growth rate

γ^*_{old}

$\dot{Y}/Y$

γ^*_{new}

time

(c)

Figure 6. Transitional dynamics (a rise in ν)

(32) with $\partial\gamma^*/\partial\nu = 0$, a rise in ν has a positive instantaneous and negative long-run effect on the equilibrium utilization rate of private capital. The equilibrium utilization rate of private capital instantaneously increases. In the next instant, it declines and converges to a new stationary value less than the initial value. Consequently, as shown in (37), the time rate of change of equilibrium utilization rate of private capital undershoots the equilibrium rate.

The transitional effect of a rise in ν upon the economic growth rate is illustrated in Figure 6(c). By Proposition 11, a rise in ν has a positive instantaneous effect on economic growth through an instantaneous effect on the growth rate of private capital stock and has negative instantaneous effects on economic growth through the instantaneous effects on the time rate of change for the equilibrium utilization rate of private capital and the growth rate of public capital. Ultimately, the instantaneous effect on the growth rate of private capital stock and on the growth rate of public capital stock are important to determine the sign of equation (38). Figure 6(c) illustrates the case where the instantaneous effect on the growth rate of private capital stock is sufficiently larger than the effect on the growth rate of public capital stock The above results can be summarized as the following proposition:

Proposition 12. *Suppose that the utilization rate of public capital is equal to the growth-maximizing equilibrium utilization rate of public capital and $\kappa < 0$. In response to an unexpected rise in the utilization rate of public capital, the time rate of change of the utilization rate of private capital undershoots the equilibrium time rate of change that equals zero. Meanwhile, the economic growth rate overshoots (undershoots) the equilibrium growth rate if the overshoot of the private capital stock growth rate is sufficiently larger (smaller) than the undershoot of the public capital stock growth rate.*

Finally, we consider the welfare effect of a change in ν. On the balanced-growth path, partial differentiation of the indirect utility function with respect to τ leads to

$$\frac{\partial W_0}{\partial \tau} = C_0^{1-\theta} \int_0^\infty \left[\frac{1}{C_0}\frac{\partial C_0}{\partial \nu} + \int_0^t \frac{\partial}{\partial \nu}\left(\frac{\dot{C}_s}{C_s}\right) ds \right] e^{-\Gamma t} dt. \tag{39}$$

Similar to equation (26), the right hand side of equation (39) is composed of the instantaneous effect on private consumption and the effect on the growth rate of private consumption.

The following proposition provides the relation between the growth-maximizing utilization rate and welfare-maximizing public capital utilization rate:

Proposition 13. *The welfare-maximizing public capital utilization rate is larger than the growth-maximizing utilization rate if $\kappa > 0$. However, the welfare-maximizing public capital utilization rate might be smaller than the growth-maximizing utilization rate if $\kappa < 0$.*

Proof. Evaluating (39) at the growth-maximizing utilization rate of public capital, we arrive at

$$\text{sgn}\left.\frac{\partial W_0}{\partial \nu}\right|_{\frac{\partial \gamma}{\partial \nu}=0} = \text{sgn}\left[\left.\frac{1}{C_0}\frac{\partial C_0}{\partial \nu}\right|_{\frac{\partial \gamma}{\partial \nu}=0} + \int_0^t \left.\frac{\partial}{\partial \nu}\left(\frac{\dot{C}_s}{C_s}\right)\right|_{\frac{\partial \gamma}{\partial \tau}=0} ds\right], \tag{40}$$

where

$$\int_0^t \left.\frac{\partial}{\partial \nu}\left(\frac{\dot{C}_s}{C_s}\right)\right|_{\frac{\partial \gamma}{\partial \nu}=0} ds = -\frac{\psi}{\phi+\psi}\frac{n^* x^* f''(x^*)}{\theta}t > 0.$$

If $\kappa > 0$, the first term in the right hand side of (40) is positive, and the sum of the terms in the right hand side of (40) are also positive. Then, we have $\partial W_0/\partial\nu > 0$ at $\nu = \bar{\nu}$. If $\kappa < 0$ is sufficiently large, the first term in the right hand side of (40) is negative and the sum of the terms in its right hand side might be negative. Thus, $\partial W_0/\partial\nu < 0$ might hold at $\nu = \bar{\nu}$.

Proposition 13 shows that a growth-maximizing government has not only an incentive for excess use of public capital, but also an incentive for its insufficient use. Note that the effect on the growth rate of private consumption is positive. According to Lemma 3, we know that a rise in ν has a positive or negative instantaneous effect on private consumption according to the sign of κ. When $\kappa < 0$, the instantaneous effect on private consumption is negative. As a result, the total welfare

effect of a rise in ν is ambiguous.

If κ is in the immediate vicinity of zero, a rise in ν has a positive welfare effect because the growth effect dominates over the instantaneous effect.　However, if κ is considerably smaller than zero, the instantaneous effect dominates over the growth effect, meaning a rise in ν has a negative welfare effect.　By (10), a higher ψ and ϕ incurs a higher possibility of $\kappa > 0$.　Therefore, as shown in the case of public investment, a high private capital utilization cost increases the possibility of a positive welfare effect of public capacity utilization.　Moreover, when private capital service complements the public capital service, a growth-maximizing government might be induced to insufficiently use public capital.

5　Conclusion

　This paper analyzed the effects of fiscal policy and public capacity utilization on economic performance and welfare.　We incorporated the capital utilization decision in an endogenous growth model with public capital.　As a result, we found that the ratio of private to public capital service is flexible in response to a change in fiscal policy and other deep parameters because the private capital utilization rate quickly reacts to any such changes.　This mutable property imparts instantaneous effects on economic variables, which is separate from models excluding capital utilization.　The degree of substitutability or complementarity between private and public capital service is important to determine the impacts of a change in the ratio of private to public capital service.

　This paper proved that maximizing capacity is equivalent to maximizing the economic growth rate in the long-run, although maximizing capacity is not equivalent to maximizing the economic growth rate in the short-run.　This result is common to two cases, one of a change in income tax rate and the other of a change in the public capacity rate.　We have also found that the growth-maximizing tax rate is increasing (decreasing) at the marginal cost of private capital utilization if the elasticity of input substitution is larger（smaller）than unity.　Welfare analysis of fiscal policy has revealed that both over-investment and under-investment in public capi-

tal stock are conceivable depending on the variables relating to the cost of capital utilization and the secondorder output elasticity.

Moreover, this paper analyzed the effects of public capacity utilization on both economic performance and welfare. We showed that the growth-maximizing utilization rate of public capital is decreasing at the marginal cost of private capital utilization, and that the public capacity utilization affects the growth-maximizing tax rate according to the elasticity of input substitution and the level of the public capacity utilization. Welfare analysis of public capacity utilization have demonstrated that both an excess use and insufficient use of public capital are conceivable depending on the variables relate to the cost of capital utilization and the second-order output elasticity.

Finally, we point out some conceivable extensions and directions for future research. In this paper, we ignored the endogenous supply of labor to keep our theoretical framework simple and focused on the presence of capital utilization. However, if our assumption of an inelastic labor supply is relaxed, then different transmission mechanisms for policy effects will be provided in the extended models. This paper also abstracted from all issues associated with alternative financial sources of public investment. Particularly, capacity and investment choice are influenced by the taxation system. These topics as well as other relevant issues will be left for future investigations.

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Appendix （Not for publication）

A.　Derivation of (6) and (7)

Equations (2), (4) and (5) provide

$$\frac{\dot{K}_t}{K_t} = \frac{(1-\tau)u_t f(x_t)}{x_t} - c_t - \delta(u_t), \tag{41}$$

$$\frac{\dot{G}_t}{G_t} = \tau v_t f(x_t) - \eta(v_t). \tag{42}$$

Using equations (41) and (42) and the definition of c_t and k_t, we obtain

$$\frac{\dot{k}_t}{k_t} = \frac{\dot{K}_t}{K_t} - \frac{\dot{G}_t}{G_t} = \frac{(1-\tau)u_t f(x_t)}{x_t} - c_t - \delta(u_t) - \tau \nu f(x_t) + \eta(\nu).$$

$$\frac{\dot{c}_t}{c_t} = \frac{\dot{C}_t}{C_t} - \frac{\dot{K}_t}{K_t} = \frac{(1-\tau)u_t f'(x_t) - \rho - \delta(u_t)}{\theta} - \frac{(1-\tau)u_t f(x_t)}{x_t} + c_t + \delta(u_t).$$

B.　Proof of Proposition 1

Existence and uniquness.　In the stationary equilibrium, $\dot{C}_t/C_t = \dot{G}_t/G_t$ holds.　Using (3) and (42) as well as $v_t = \nu$ and $\dot{C}_t/C_t = \dot{G}_t/G_t$, we obtain

$$\frac{(1-\tau)u_t f'(x_t) - \rho - \delta(u_t)}{\theta} = \tau \nu f(x_t) - \eta(\nu) \tag{43}$$

The left hand side of (43) decreases with respect to x because

$$\frac{\partial}{\partial x_t}\left(\frac{\dot{C}_t}{C_t}\right) = \left[(1-\tau)\left\{\frac{\partial u_t}{\partial x_t}f'(x_t) + u_t f''(x_t)\right\} - \frac{\partial u_t}{\partial x_t}\delta'(u_t)\right]\theta^{-1}$$

$$= \frac{(1-\tau)u_t f''(x_t)}{\theta} < 0$$

where

$$\frac{\partial u_t}{\partial x_t} = \frac{(1-\tau)f''(x_t)}{\delta''(u_t)} < 0.$$

Furthermore, the right hand side of ⑷ increases with respect to x since

$$\frac{\partial}{\partial x_t}\left(\frac{\dot{G}_t}{G_t}\right) = \tau\nu f'(x_t) > 0.$$

Note that

$$\lim_{x_t \to 0}\frac{(1-\tau)u_t f'(x_t) - \rho - \delta(u_t)}{\theta} > 0 > \lim_{x_t \to 0}\tau\nu f(x_t) - \eta(\nu),$$

$$\lim_{x_t \to \infty}\frac{(1-\tau)u_t f'(x_t) - \rho - \delta(u_t)}{\theta} < 0 < \lim_{x_t \to \infty}\tau\nu f(x_t) - \eta(\nu).$$

These conditions show that a unique x^* exists that satisfies ⑷. Since $x_t = x(k_t, \nu, \tau)$ is monotonically increasing with respect to k_t, x^* gives a unique value k^* such that $x^* = x(k^*, \nu, \tau)$. Then, we have

$$c^* = \frac{(1-\tau)u^* f(x^*)}{x^*} - \delta(u^*) - \gamma^*,$$

$$\gamma^* = \frac{(1-\tau)u^* f'(x^*) - \rho - \delta(u^*)}{\theta}.$$

These equations then lead to

$$c^* > 0 \Leftrightarrow \frac{(1-\tau)u^* f(x^*)}{x^*} - \delta(u^*) > \gamma^*,$$

$$\gamma^* > 0 \Leftrightarrow \frac{(1-\tau)u^* f'(x^*) - \rho - \delta(u^*)}{\theta} = \frac{u^*\delta'(u^*) - \delta(u^*) - \rho}{\theta} > 0.$$

To satisfy $c^* > 0$ and $\gamma^* > 0$, we need

$$\frac{(1-\tau)u^* f(x^*) - x^*\delta(u^*)}{x^*} > \frac{u^*\delta'(u^*) - \delta(u^*) - \rho}{\theta} > 0.$$

Stability. The linearized system of ⑹ and ⑺ is

$$\begin{pmatrix} \dot{c}_t \\ \dot{k}_t \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}\begin{pmatrix} c_t - c^* \\ k_t - k^* \end{pmatrix}, \tag{44}$$

where

$$J_{11} = \left[\frac{\partial}{\partial c_t}\left(\frac{\dot{C}_t}{C_t}\right)\bigg|_{\dot{c}=\dot{k}=0} - \frac{\partial}{\partial c_t}\left(\frac{\dot{K}_t}{K_t}\right)\bigg|_{\dot{c}=\dot{k}=0} \right] c^* = c^* > 0,$$

$$J_{12} = \left[\frac{\partial}{\partial k_t}\left(\frac{\dot{C}_t}{C_t}\right)\bigg|_{\dot{c}=\dot{k}=0} - \frac{\partial}{\partial k_t}\left(\frac{\dot{K}_t}{K_t}\right)\bigg|_{\dot{c}=\dot{k}=0} \right] c^* \gtreqless 0,$$

$$J_{21} = \left[\frac{\partial}{\partial c_t}\left(\frac{\dot{K}_t}{K_t}\right)\bigg|_{\dot{c}=\dot{k}=0} - \frac{\partial}{\partial c_t}\left(\frac{\dot{G}_t}{G_t}\right)\bigg|_{\dot{c}=\dot{k}=0} \right] k^* = -k^* < 0,$$

$$J_{22} = \left[\frac{\partial}{\partial k_t}\left(\frac{\dot{K}_t}{K_t}\right)\bigg|_{\dot{c}=\dot{k}=0} - \frac{\partial}{\partial k_t}\left(\frac{\dot{G}_t}{G_t}\right)\bigg|_{\dot{c}=\dot{k}=0} \right] k^* < 0.$$

Note that we have

$$\frac{\partial}{\partial k_t}\left(\frac{\dot{C}_t}{C_t}\right) = \frac{(1-\tau)u_t f''(x_t)}{\theta}\frac{\partial x_t}{\partial k_t} < 0, \quad \frac{\partial}{\partial k_t}\left(\frac{\dot{G}_t}{G_t}\right) = \tau\nu f'(x_t)\frac{\partial x_t}{\partial k_t} > 0,$$

$$\frac{\partial}{\partial k_t}\left(\frac{\dot{K}_t}{K_t}\right) = -(1-\tau)\frac{\phi(x_t)+\psi(x_t)}{\psi(x_t)}\frac{u_t\alpha(x_t)f(x_t)}{x_t^2}\frac{\partial x_t}{\partial k_t} < 0,$$

$$\frac{\partial}{\partial c_t}\left(\frac{\dot{K}_t}{K_t}\right) = -1, \quad \frac{\partial}{\partial c_t}\left(\frac{\dot{C}_t}{C_t}\right) = \frac{\partial}{\partial c_t}\left(\frac{\dot{G}_t}{G_t}\right) = 0.$$

The characteristic polynomial is given as $P(\xi) = 0$ where $P(\xi) := \xi^2 - \mathrm{trace}J\xi + \det J = 0$. Since we have

$$\det J = J_{11}J_{22} - J_{12}J_{21} = \left[\frac{\partial}{\partial k_t}\left(\frac{\dot{C}_t}{C_t}\right)\bigg|_{\dot{c}=\dot{k}=0} - \frac{\partial}{\partial k_t}\left(\frac{\dot{G}_t}{G_t}\right)\bigg|_{\dot{c}=\dot{k}=0} \right] c^*k^* < 0,$$

the characteristic polynomial has one positive and one negative root. The dynamic system in this model has one state k_t and one control variable c_t for a given ν and τ. Therefore, the stationary equilibrium is stable in the saddle–point sense.

The property of γ^*. The partial differentiation of γ^* with respect to u^* yeilds

$$\frac{\partial\gamma^*}{\partial u^*} = \frac{u^*\delta''(u^*)}{\theta} > 0.$$

This equation shows that γ^* is monotonically increasing in u^*.

C. Proof of Lemma 1

The general solution to the linearized system of equations (6) and (7) is

$$
\begin{pmatrix} c_t - c^* \\ k_t - k^* \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} e^{\lambda t} \\ e^{\mu t} \end{pmatrix}
\tag{45}
$$

In equation (45), A_{ij} is the vector for arbitrary constants $(i, j = 1, 2)$. Let λ be a negative eigenvalue, and μ be a positive eigenvalue. Since k_t is a state variable (not jumpable), we have $A_{22}=0$. At time $t=0$, $A_{22}=0$ and $k_t - k^* = A_{21}e^{\lambda t} + A_{22}e^{\mu t}$ engender $A_{21}=k_0-k^*$. Therefore, we obtain

$$
k_t = k^* + (k_0 - k^*)e^{\lambda t}.
$$

This is equation (9) in the maintext.

Differentiating equation (9) with respect to t yields

$$
\dot{k}_t = (k_0 - k^*)\lambda e^{\lambda t}
\tag{46}
$$

Using equations (44), (45), and (46) with $A_{22} = 0$, we obtain

$$
\dot{k}_t = J_{21}[A_{11}e^{\lambda t} + A_{12}e^{\mu t}] + (k_0 - k^*)J_{22}e^{\lambda t}
\tag{47}
$$

Comparing equations (46) and (47), the vectors A_{ij} become

$$
A_{12} = A_{22} = 0
\tag{48}
$$

$$
(k_0 - k^*)\lambda = J_{21}A_{11} + (k_0 - k^*)J_{22} \Leftrightarrow A_{11} = \frac{(k_0 - k)(\lambda - J_{22})}{J_{21}}
\tag{49}
$$

Using equations (45), (48) and (49), we obtain $c_t = c^* + (k_0 - k^*)\kappa e^{\lambda t}$ where $\kappa := (\lambda - J_{22})/J_{21}$. This equation is just equal to equation (8). We have $P(\lambda) = 0$ and $P(J_{22}) = J_{22}^2 - (J_{11}+J_{22})J_{22} + J_{11}J_{22} - J_{12}J_{21} = -J_{12}J_{21}$. If $J_{12} < 0$, we have $P(J_{22}) < 0$, i.e. $\lambda < J_{22} < 0$. Therefore, κ should be a positive. When $J_{12} > 0$, we obtain $P(J_{22}) > 0$, i.e. $J_{22} < \lambda < 0$.

Accordingly, κ should be a negative.　J_{12} is

$$J_{12} = \left[\alpha^* - \frac{(1-\alpha^*)\phi^*\psi^*}{(\phi^*+\psi^*)\theta} \right] \frac{(1-\tau)u^*c^*f(x^*)}{x^*k^*} \gtreqless 0$$

$$\Leftrightarrow \theta \gtreqless \left(\frac{1-\alpha^*}{\alpha^*} \right) \left(\frac{\phi^*\psi^*}{\phi^*+\psi^*} \right).$$

Therefore, by abstracting the asterisk of α^*, ϕ^* and ψ^*, we have

$$\kappa \lesseqgtr 0 \Leftrightarrow \theta \gtreqless \left(\frac{1-\alpha}{\alpha} \right) \left(\frac{\phi\psi}{\phi+\psi} \right).$$

D.　Proof of Lemmas 3 and 4

Derivation of ⑲, ⑳, ㉜, *and* ㉝ *in Lemma 3 and 4.*　By the properties of $u_t = u(k_t, \nu, \tau)$ and $x_t = x(k_t, \nu, \tau)$, we obtain

$$\frac{\partial u_0}{\partial \tau} = -\frac{\nu f'(x_0)}{\nu \delta''(u_0) - k_0(1-\tau)f''(x_0)} < 0 \Rightarrow \frac{\tau}{u_0}\frac{\partial u_0}{\partial \tau} = -\frac{\tau}{(1-\tau)(\phi_0+\psi_0)} < 0,$$

$$\frac{\partial x_0}{\partial \tau} = k_0\frac{\partial n_0}{\partial \tau} = \frac{k_0}{\nu}\frac{\partial u_0}{\partial \tau} < 0 \Rightarrow \frac{\tau}{x_0}\frac{\partial x_0}{\partial \tau} = \frac{\tau}{u_0}\frac{\partial u_0}{\partial \tau},$$

$$\frac{\partial u_0}{\partial \nu} = -\frac{(1-\tau)x_0 f''(x_0)}{\nu \delta''(u_0) - k_0(1-\tau)f''(x_0)} > 0 \Rightarrow 0 < \frac{\nu}{u_0}\frac{\partial u_0}{\partial \nu} = \frac{\phi_0}{\phi_0+\psi_0} < 1,$$

$$\frac{\partial x_0}{\partial \nu} = \frac{k_0}{\nu}\frac{\partial u_0}{\partial \nu} - \frac{u_0 k_0}{\nu^2} \Rightarrow \frac{\nu}{x_0}\frac{\partial x_0}{\partial \nu} = \frac{\nu}{u_0}\frac{\partial u_0}{\partial \nu} - 1 < 0$$

Equations ⑱–⑳ and ㉛–㉝ are derived from these equations.

　　Comparative dynamics.　See Judd (1982) for the details of the method presented in the remainder of this Appendix.　Differentiating equations ⑥ and ⑦ with respect to τ around the stationary equilibrium, we obtain

$$\begin{pmatrix} \frac{\partial \dot{c}_t}{\partial \tau} & \frac{\partial \dot{c}_t}{\partial \nu} \\ \frac{\partial \dot{k}_t}{\partial \tau} & \frac{\partial \dot{k}_t}{\partial \nu} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial c_t}{\partial \tau} & \frac{\partial c_t}{\partial \nu} \\ \frac{\partial k_t}{\partial \tau} & \frac{\partial k_t}{\partial \nu} \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{50}$$

where

$$B_{11} = \left[\theta - \frac{(1-\alpha^*)\psi^*}{\phi^* + \psi^*}\right] \frac{c^* u^* f(x^*)}{\theta x^*} \gtreqless 0$$

$$B_{12} = -\left[\theta - \left(\frac{1-\alpha^*}{\alpha^*}\right)\left(\frac{\phi^*\psi^*}{\phi^* + \psi^*}\right)\right] \frac{(1-\tau)\alpha^* c^* n^* f(x^*)}{\theta x^*} \gtreqless 0$$

$$B_{21} = -\left[1 + \frac{(1-\tau)(\phi^* + \psi^*) - (1-\alpha^*)\tau}{(1-\tau)(\phi^* + \psi^*)}k^*\right] \frac{k^* u^* f(x^*)}{x^*} \gtreqless 0$$

$$B_{22} = \left[\frac{(1-\tau)\alpha^* n^* f(x^*)}{x^*} - \tau f(x^*) + \eta'(\nu)\right] k^* \gtreqless 0$$

The dynamic system in ⑸ will have a unique bounded solution because it has a negative eigenvalue $\lambda < 0$ and a positive eigenvalue $\mu > 0$. Using the Laplace transformation of c_t and k_t, i.e., $Q_t := \int_0^\infty c_t e^{-\chi t} dt$ and $R_t := \int_0^\infty k_t e^{-\chi t} dt$ where χ is a positive constant, equation ⑸ becomes

$$\begin{pmatrix} \frac{\partial Q_t}{\partial \tau} & \frac{\partial Q_t}{\partial \nu} \\ \frac{\partial R_t}{\partial \tau} & \frac{\partial R_t}{\partial \nu} \end{pmatrix} = (\chi I - J)^{-1} \begin{pmatrix} \frac{\partial c_0}{\partial \tau} + \frac{B_{11}}{\chi} & \frac{\partial c_0}{\partial \nu} + \frac{B_{12}}{\chi} \\ \frac{B_{21}}{\chi} & \frac{B_{22}}{\chi} \end{pmatrix} \tag{51}$$

where I is the identity matrix at 2×2. Note that we have $\partial s_0/\partial \tau = 0$.

Using equation ⑸ and $\chi = \mu$, we obtain

$$(\mu - J_{22})\left[\frac{\partial c_0}{\partial \tau} + \frac{B_{11}}{\mu}\right] + \frac{J_{12}B_{21}}{\mu} = 0, \quad J_{21}\left[\frac{\partial c_0}{\partial \tau} + \frac{B_{11}}{\mu}\right] + \frac{(\mu - J_{11})B_{21}}{\mu} = 0$$

$$(\mu - J_{22})\left[\frac{\partial c_0}{\partial \nu} + \frac{B_{12}}{\mu}\right] + \frac{J_{12}B_{22}}{\mu} = 0, \quad J_{21}\left[\frac{\partial c_0}{\partial \nu} + \frac{B_{12}}{\mu}\right] + \frac{(\mu - J_{11})B_{22}}{\mu} = 0$$

After some manipulations, we arrive at

$$\frac{\partial c_0}{\partial \tau} = -\frac{B_{11}}{\mu} - \frac{(\mu - J_{11})B_{21}}{\mu J_{21}} \text{ and } \frac{\partial c_0}{\partial \nu} = -\frac{B_{12}}{\mu} - \frac{(\mu - J_{11})B_{22}}{\mu J_{21}}$$

Since $c_0 = C_0/K_0$, we have $\partial C_0/\partial \tau = K_0 \partial c_0/\partial \tau$ and $\partial C_0/\partial \nu = K_0 \partial c_0/\partial \nu$.

Derivation of ⒅ *in Lemma 3*. Note that $B_{11} > 0$ if $\theta \geq 1$. Evaluating $\partial c_0/\partial \tau$ at $\partial \gamma/\partial \tau = 0$ and abstracting the asterisk of α^*, ϕ^* and ψ^*, we have

$$\left.\frac{\partial c_0}{\partial \tau}\right|_{\frac{\partial \gamma}{\partial \tau}=0} = -\frac{B_{11}}{\mu}\left[1 - \frac{c^* - \mu}{c^*}\frac{(\phi + \psi)\theta + \psi k^*}{(\phi + \psi)\theta - (1 - \alpha)\psi}\right]$$

Evaluating $\partial c_0/\partial \nu$ at $\partial \gamma/\partial \nu = 0$ and abstracting the asterisk of α^*, ϕ^* and ψ^*, we obtain

$$\left.\frac{\partial c_0}{\partial \nu}\right|_{\frac{\partial \gamma}{\partial \nu}=0} = -\frac{B_{12}}{\mu} \gtreqless 0 \Leftrightarrow \kappa$$